Universal characteristics of statistically steady state for integrable turbulence developing from partially incoherent wave initial conditions.

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Introduction.

We study numerically statistics of waves for 1D focusing Nonlinear Schrodinger Equation,

$$i\psi_t + \psi_{xx} + |\psi|^2 \psi = 0.$$

We use is Runge-Kutta 4th-order method on adaptive grid combined with Fourier interpolation. This method conserves very well the first 12 integrals of motion (with error smaller than 10^{-6}), including

wave action $$N = \frac{1}{L}\int_{-L/2}^{L/2} |\psi(x,t)|^2\, dx = \sum_k |\psi_k|^2,$$

momentum $$P = \frac{i}{2L}\int_{-L/2}^{L/2} (\psi_x^*\psi - \psi_x\psi^*)\, dx = \sum_k k\, |\psi_k|^2,$$

total energy $$E = H_2 + H_4, \quad H_d = \frac{1}{L}\int_{-L/2}^{L/2} |\psi_x|^2\, dx, \quad H_4 = -\frac{1}{2L}\int_{-L/2}^{L/2} |\psi|^4\, dx.$$

Introduction.

We start from Random Phase initial field,

$$\psi = \sum_k |\psi_k| e^{ikx+i\phi_k}, \quad |\psi_k|^2 = S_k,$$

with average square amplitude (in x-space) equal to unity,

$$\overline{|\psi|^2} = 1.$$

Here $\mathbf{S_k}$ is a smooth function and φ_k are random uncorrelated phases. From the central limit theorem we know that, for such initial conditions, the probability density function (PDF) of wave amplitude is Rayleigh distribution,

$$P_R(|\psi|) = 2|\psi| e^{-|\psi|^2},$$

the kurtosis is equal to two,

$$\kappa_0 = \frac{\langle|\psi|^4\rangle}{\langle|\psi|^2\rangle^2} = 2,$$

and, thus, the ensemble-average potential energy is minus one,

$$\langle H_4 \rangle_0 = -1.$$

Initial kinetic energy can be calculated through initial wave-action spectrum,

$$\langle H_d \rangle_0 = \sum_k k^2 S_k.$$

We use sufficiently large boxes **L** (~ 256π) in order to make all standard deviations small.

Introduction.

For each initial spectrum $\mathbf{S_k}$, we generate ensemble of initial conditions containing 1000 independent realizations and simulate their evolution using the NLS equation until time **T~50** when ensemble-averaged statistical characteristics (like wave-action spectrum and the PDF) do not change any more (statistically steady state). In this state we additionally average these statistical characteristics over time.

We pay special attention to ensemble-averaged kinetic and potential energies and to the PDF of wave amplitude. Note that if the PDF of wave amplitude is Rayleigh function, then the PDF of square amplitude is exponential,

$$P_R(|\psi|^2) = e^{-|\psi|^2}.$$

For convenience, we will call this exponential distribution as Rayleigh PDF too.

Results.

We start from Gaussian initial spectrum,

$$S_k = A_0 e^{-2k^2/\theta^2}.$$

Here θ determines initial spectral width and constant A_0 is obtained from normalization

$$\overline{|\psi|^2} = 1.$$

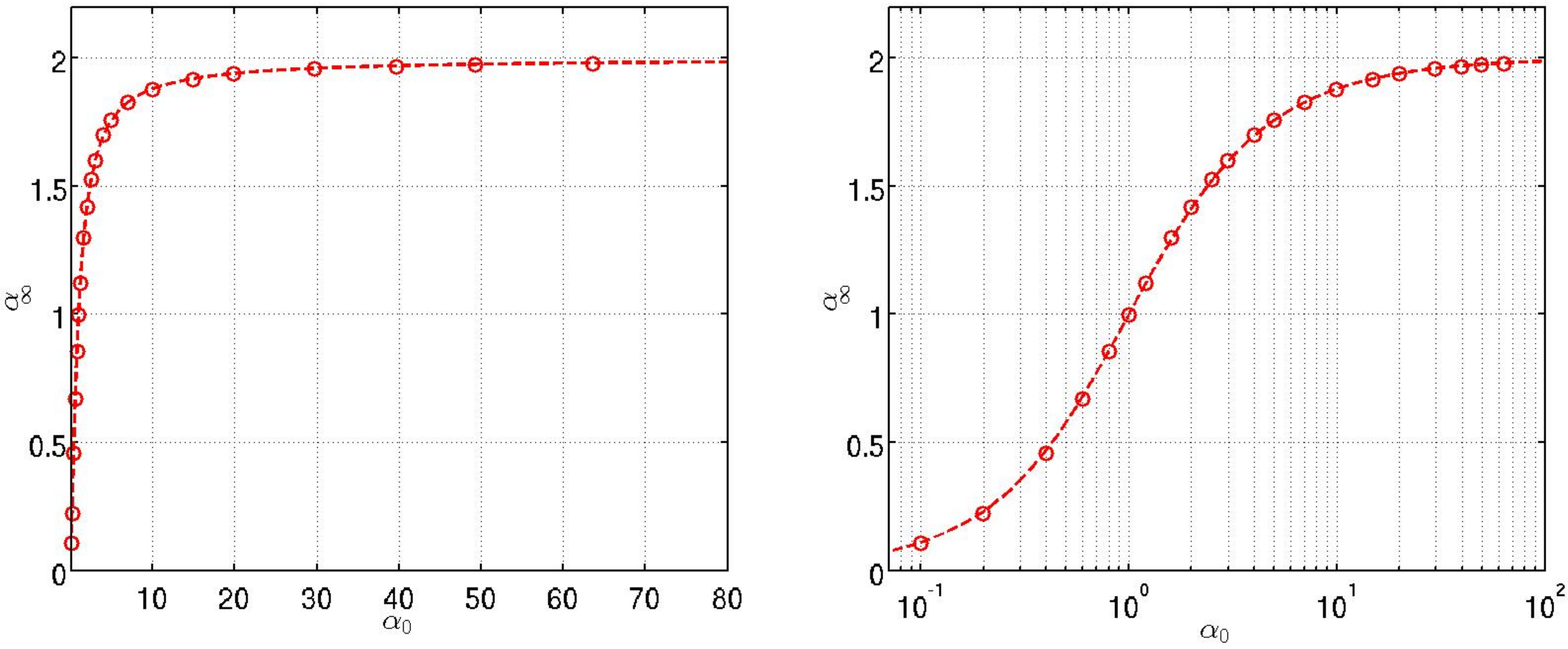
As can be easily found, initial potential to kinetic energy ratio is

$$\alpha_0 = \frac{|\langle H_4 \rangle|}{\langle H_d \rangle} = \frac{6\Gamma_{3/2}}{\Gamma_{5/2}} \times \frac{1}{\theta^2},$$

where Γ is gamma-function. If we start from super-Gaussian spectrum, then the dependence of the initial ratio on spectral width θ is the same, but the constant coefficient is different.

How the energy ratio in the statistically steady state depends on the initial one?

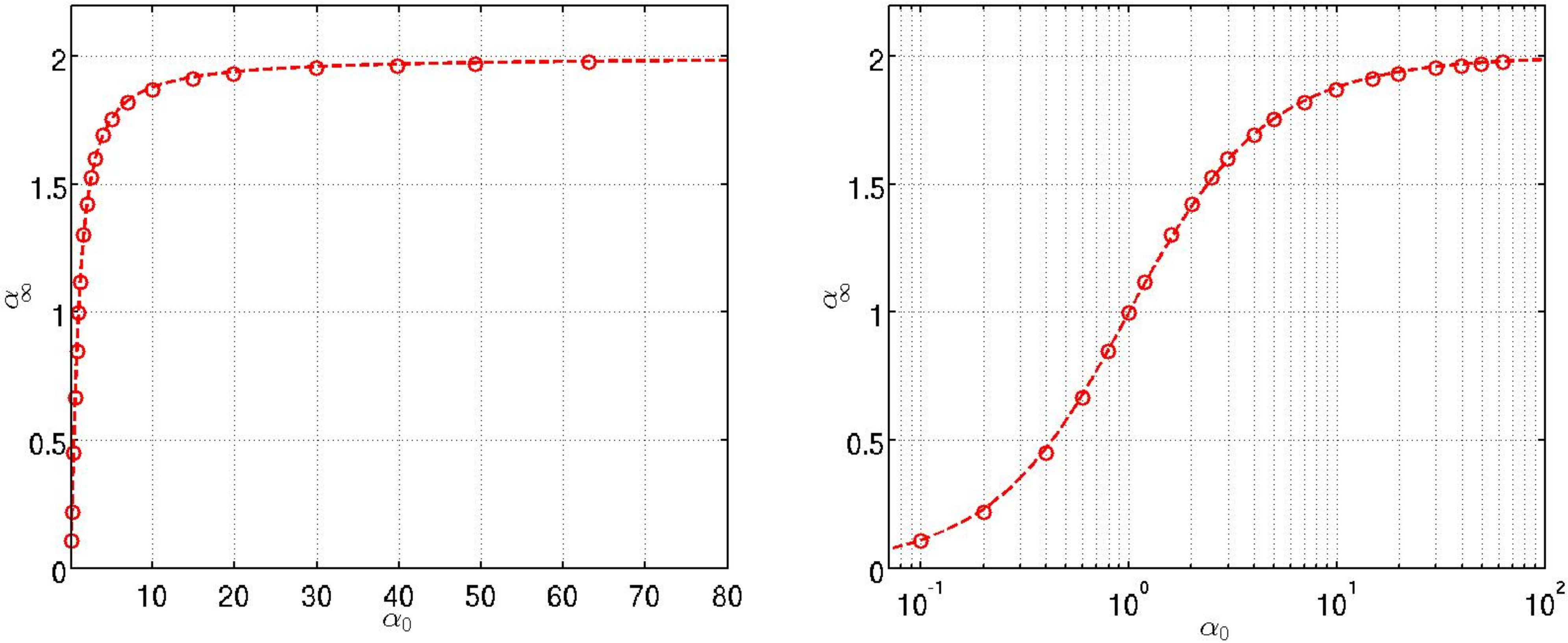
Results.



Final energy ratio increases monotonically with the initial one, from zero to two, where it has asymptotic limits.

Note that, from the energy conservation law, if the initial ratio is equal to one, then the final one is unity too.

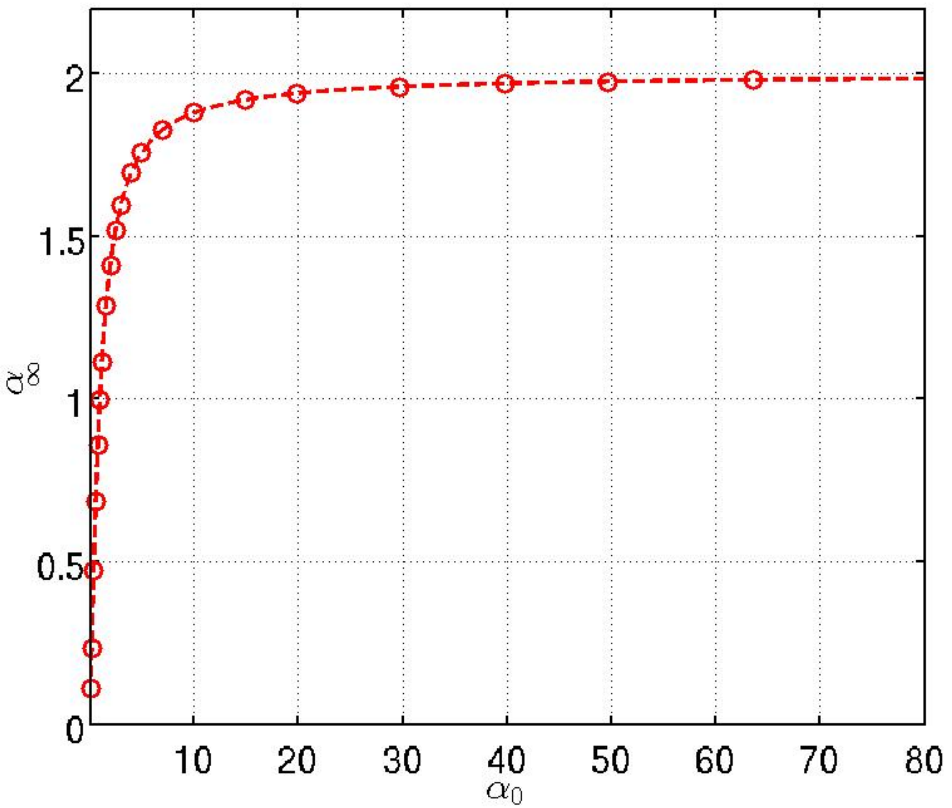
Results.

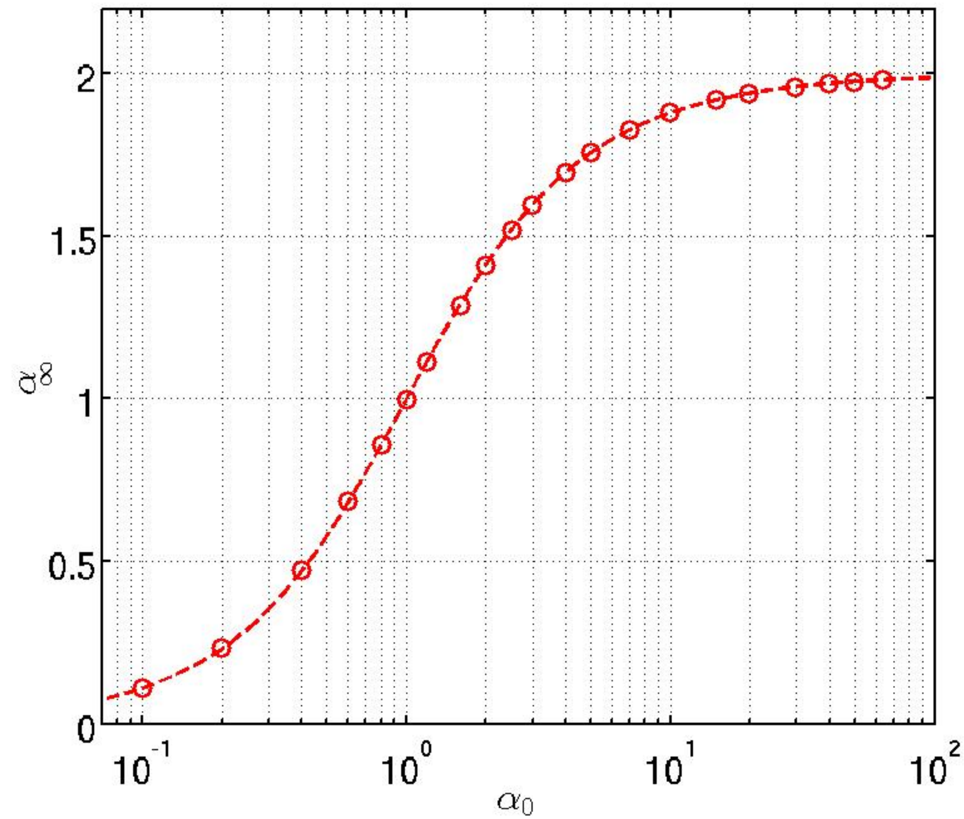


Same results for super-Gaussian spectrum,

$$S_k = A_0 e^{-2k^{32}/\theta^{32}}.$$

Results.



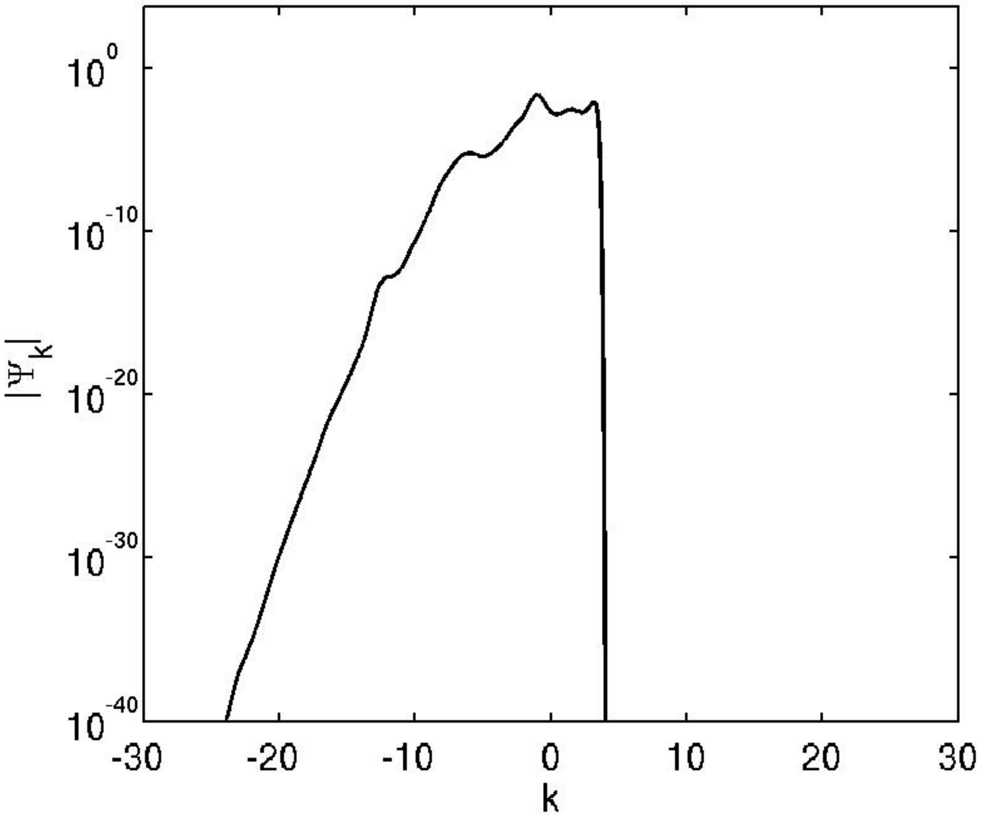


Same results for exponential spectrum,

$$S_k = A_0 e^{-2|k|/\theta}.$$

Results.

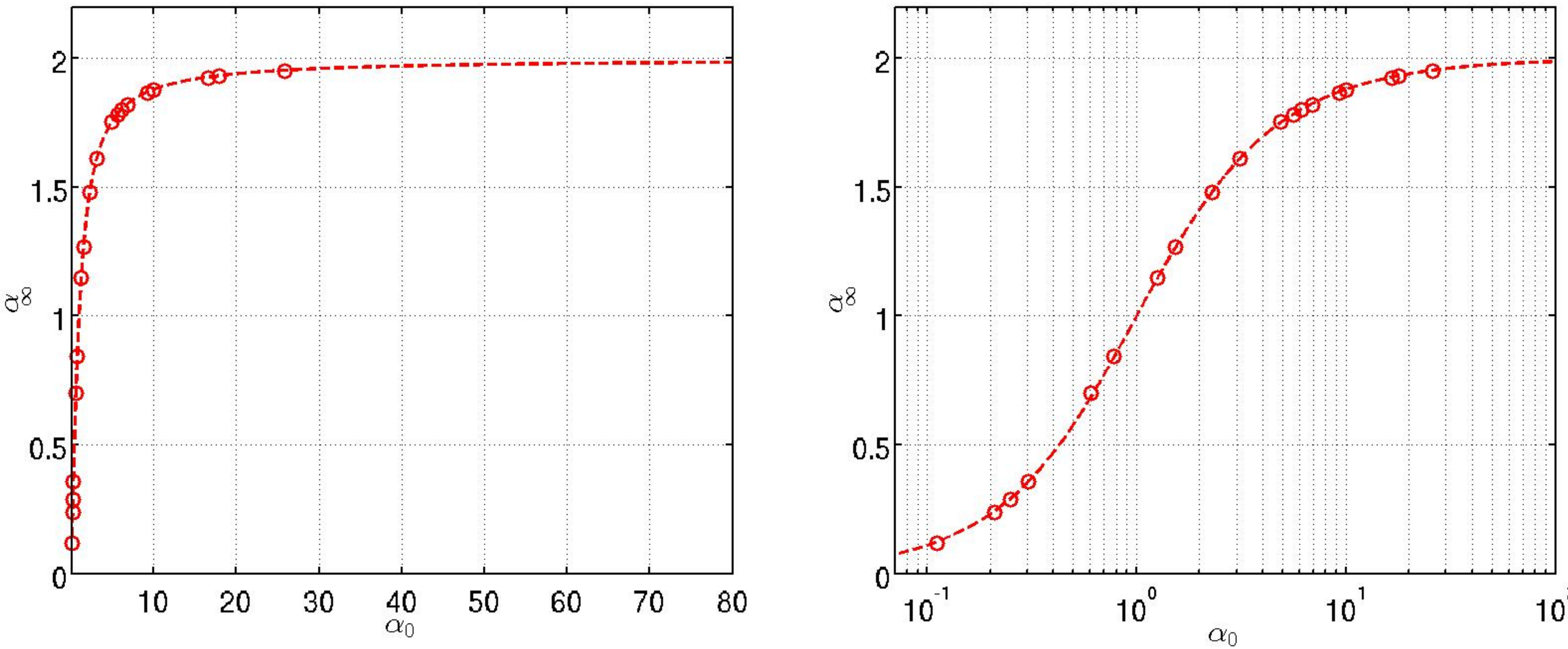
Now let's start from arbitrary non-symmetric initial spectrum, i.e. when functions S_k look like this one:



Note that total momentum must be zeroth,

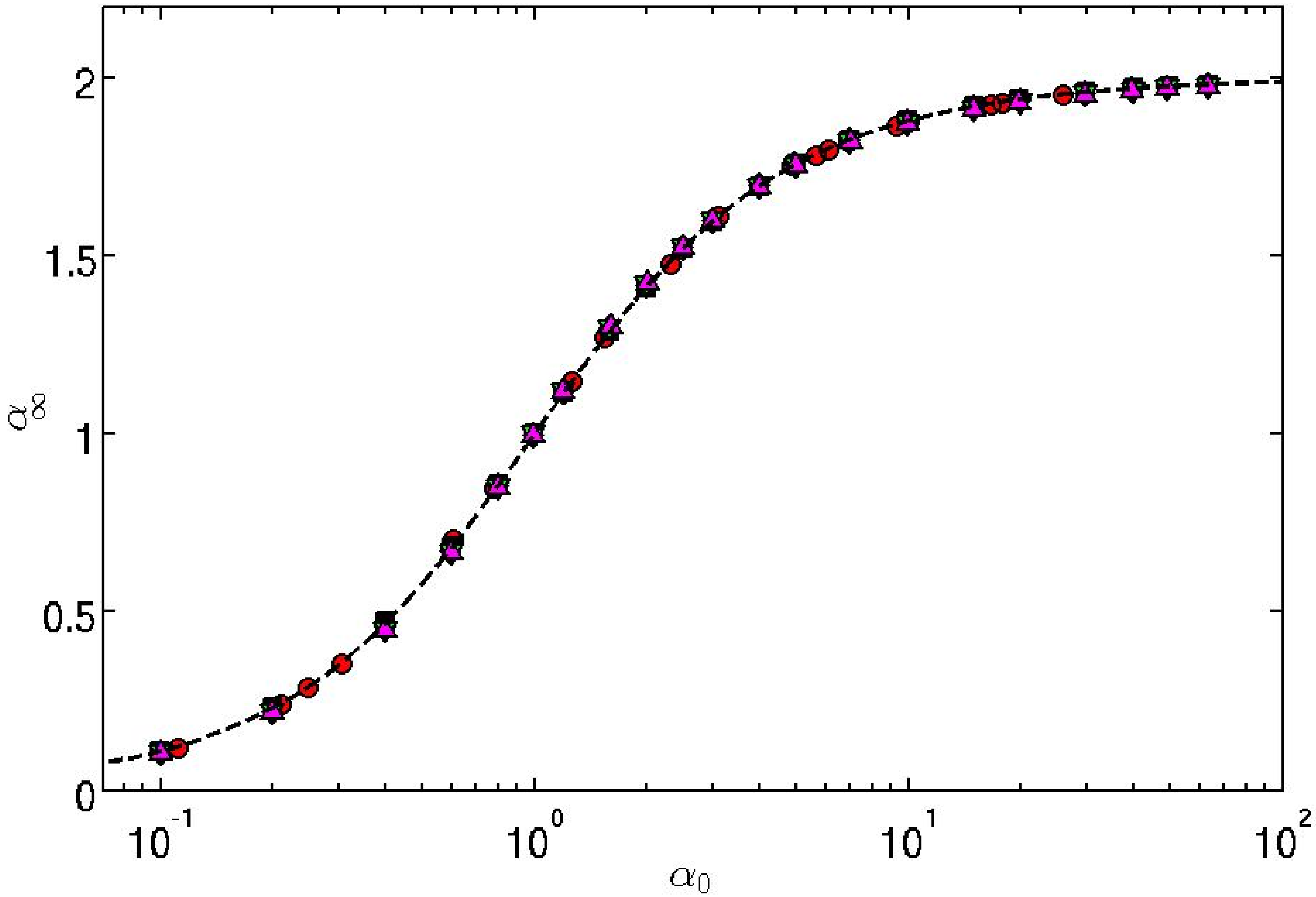
$$\sum_k kS_k = 0.$$

Results.



α_∞
α_0
0
0.5
1
1.5
2
10
20
30
40
50
60
70
80
10^{-1}
10^{0}
10^{1}
10^{2}

Results.

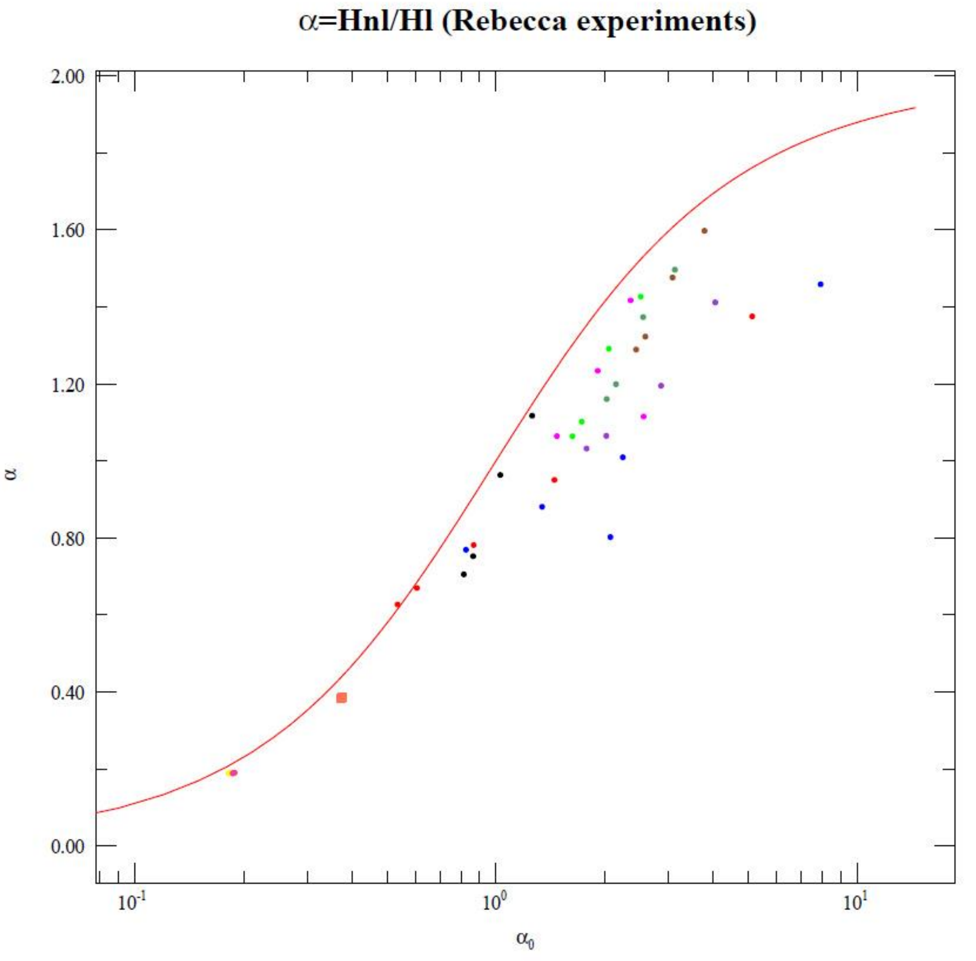


Fit for the curve:

$$\alpha_\infty = \frac{2\alpha_0^2 + b\,\alpha_0}{\alpha_0^2 + \alpha_0 + b}, \quad b \approx 0.82.$$

Numerical value of b is close to either $\sqrt{2/3}$ or $2(\sqrt{2}-1)$.

Results.



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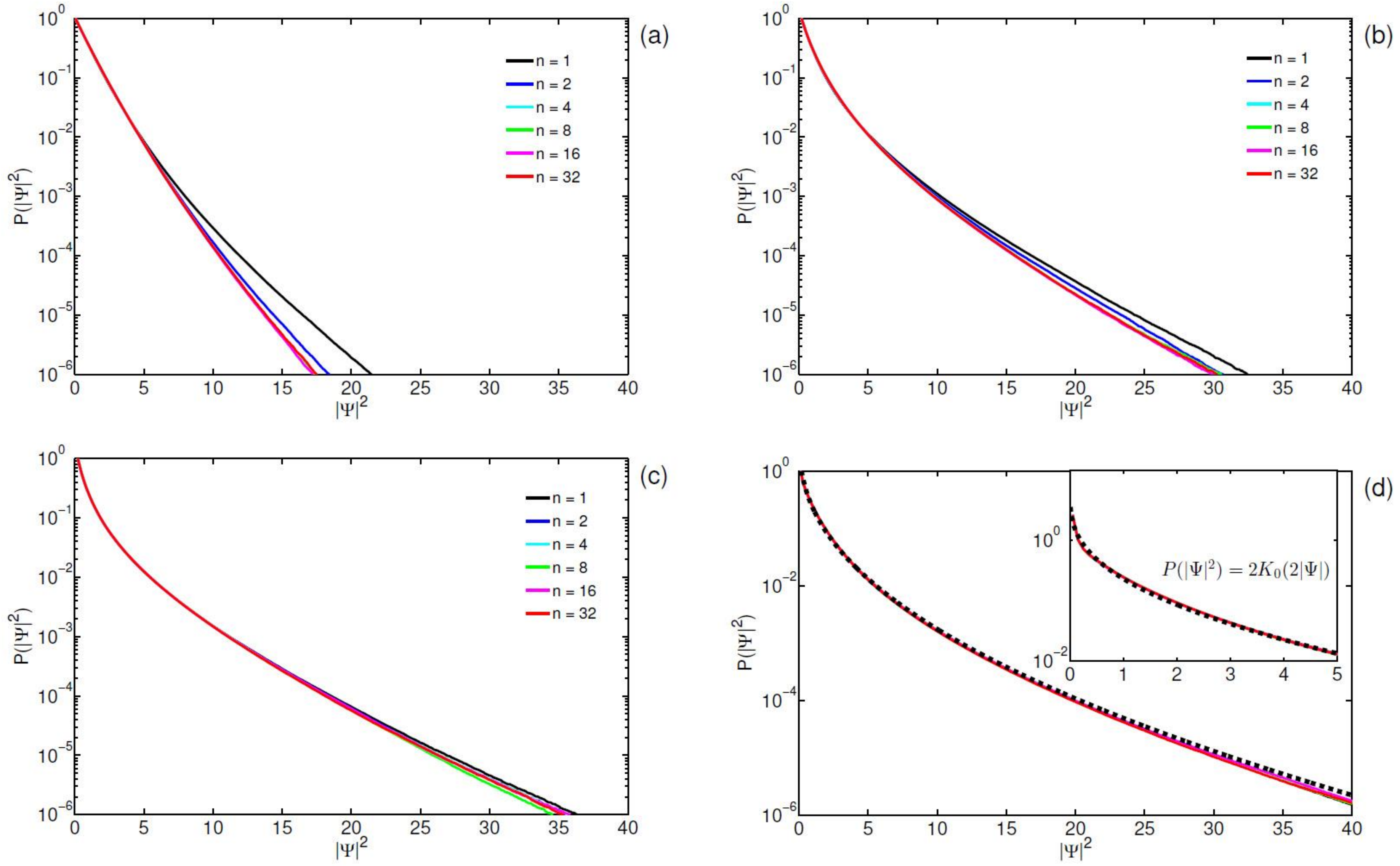


Note that statistically steady states are different! Stationary PDFs for fixed initial energy ratio of 0.1 (a), 0.8 (b), 2.5 (c) and 64 (d). We also have a universal limiting PDF for very large initial nonlinearity.

Results.

What this result for the universal energy ratio curve means?

Initial state is defined from the CLT.

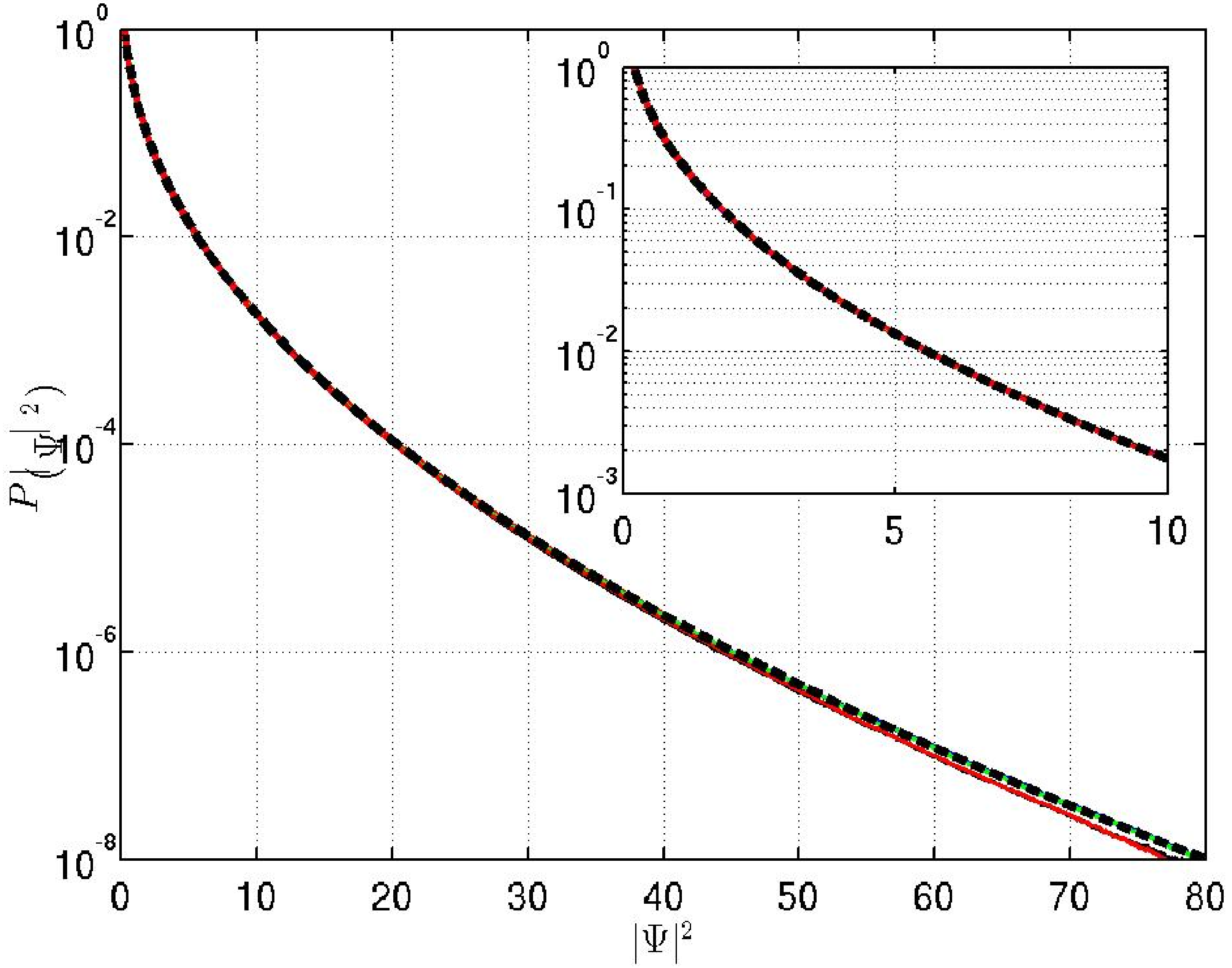
Final state is determined as statistically steady one.

Therefore, if we look only at the energy ratios, there is a universal transition from the CLT state to the statistically steady one. This transition allows one to determine final kinetic and potential energies, and also the kurtosis:

$$\langle H_d \rangle_\infty = \frac{\alpha_0^2 + \alpha_0 + b}{\alpha_0^2 + b\,\alpha_0}, \quad \langle H_4 \rangle_\infty = -\frac{2\alpha_0^2 + b\,\alpha_0}{\alpha_0^2 + b\,\alpha_0},$$

$$\kappa_\infty = 2\,|\,\langle H_4 \rangle_\infty\,| = 2\,\frac{2\alpha_0^2 + b\,\alpha_0}{\alpha_0^2 + b\,\alpha_0}.$$

Results.



PDFs at statistically steady states obtained from random phase fields with very large nonlinearity.

Results.

Let us start from i.c. with very large potential to kinetic energy ratio $\boldsymbol{\alpha_0}$. Their spectral width is very small, and we may think of such a signal as a superposition of elementary condensates of different amplitudes. After long evolution inside the fiber, each of these ``condensates'' may provide Rayleigh PDF

$$P(|\psi|^2) = I^{-1} e^{-|\psi|^2/I},$$

where **I** is the initial intensity of this "condensate", which itself has Rayleigh statistics. Then, we can guess that the resulting PDF may be constructed as arithmetic superposition of these Rayleigh PDFs coming from each element, i.e.

$$P_{IW}(|\psi|^2) = \int_0^{+\infty} I^{-1} e^{-|\psi|^2/I} \times e^{-I}\, dI = 2K_0(2|\psi|),$$

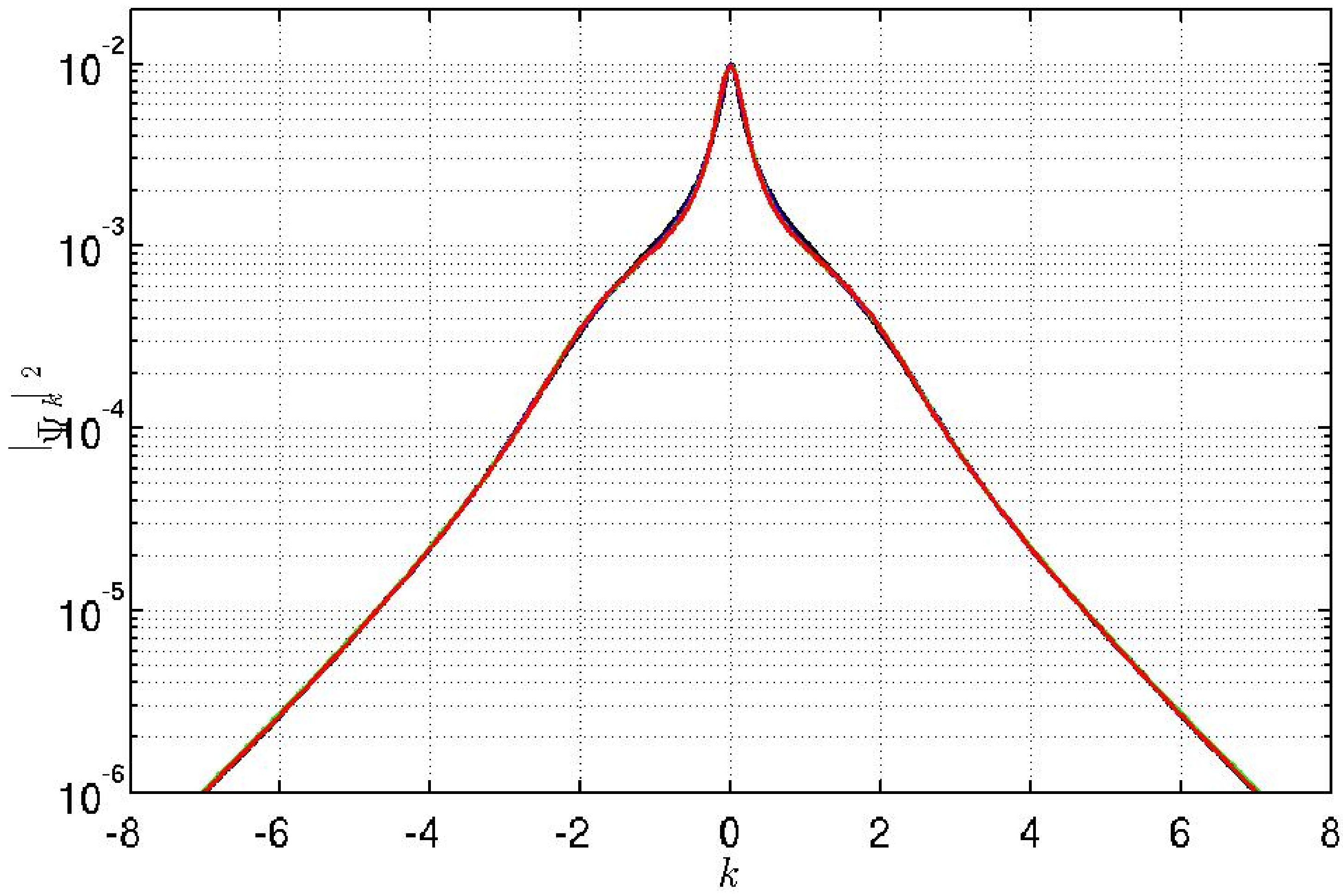
where K_0 is the modified Bessel function of the second kind. The tail of this "limiting" PDF is strongly non-Rayleigh one,

$$P_{IW}(|\psi|^2) \to \sqrt{\frac{\pi}{|\psi|}}\, e^{-2|\psi|}.$$

Together with the energy conservation law, this PDF can be used to calculate characteristics of the "limiting" statistically steady state,

$$\langle H_4 \rangle_\infty = -2, \quad \langle H_d \rangle_\infty = 1, \quad \alpha_\infty = 2, \quad \kappa_\infty = 4.$$

Results.



Limiting wave-action spectrum $<|\psi_k|^2>$ at the statistically steady state also seems to be universal. However, its shape is much more difficult to guess.

Thank you for your attention!