Effectivization of the finite-gap formulas for the self-focusing NLS equation

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P.G. Grinevich, P.M. Santini “The finite gap method and the analytic description of the exact rogue wave recurrence in the periodic NLS Cauchy problem. 1”, arXiv:1707.05659

P.G. Grinevich, P.M. Santini “Numerical instability of the Akhmediev breather and a finite-gap model of it”, arXiv:1708.00762

P.G. Grinevich, P.M. Santini “The exact rogue wave recurrence in the NLS periodic setting via matched asymptotic expansions, for 1 and 2 unstable modes”, arXiv:1708.04535

Rogue waves – large amplitude waves suddenly appear. Rogue waves in the ocean my be very dangerous.

Linear theory could not explain this phenomenon. It is very likely to be connected with modulation instability.

Self-focusing Nonlinear Schrödinger equation is used as physical model. **It is a completely integrable equation.**

Also rogue waves in nonlinear optics, Bose-condensate.

Our aim is to obtain simple approximate formulas for periodic problem under assumption that at $t=0$ we have a small perturbation of the constant solution.

$$u(x,0) = 1 + \epsilon(x), \ \ \epsilon(x+L) = \epsilon(x), \ \ \epsilon(x) \ll 1.$$

Unperturbed solution

$$u_0(x,t) = e^{2it}.$$

Consider a generic periodic perturbation:

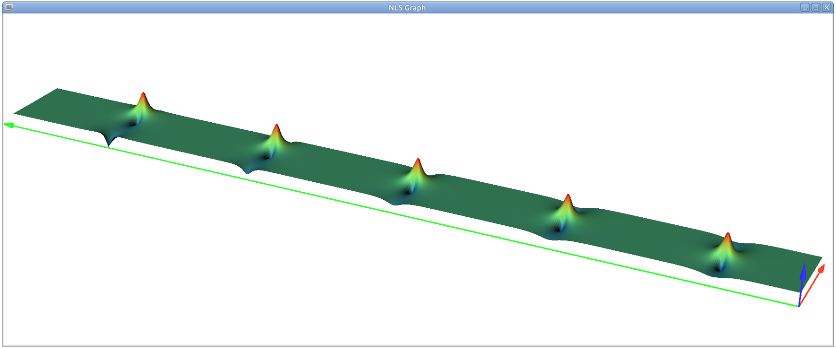
$$\epsilon(x) = \sum_{j\geq 1}\left(c_j e^{ik_j x} + c_{-j} e^{-ik_j x}\right), \;\; k_j = \frac{2\pi}{L} j, \;\; |c_j| = O(\epsilon),$$

The first N harmonics are unstable, where

$$N = \left[\frac{L}{\pi}\right]$$

with the growing factor in the linear mode:

$$\sigma_j = k_j\sqrt{4-k_j^2}, \;\; 1 \leq j \leq N,$$



Recurrence of Akhmediev breathers for one unstable mode ($L = 6$).

We study self-focusing NLS equation (SfNLS):

$$iu_t + u_{xx} + 2u^2\bar{u} = 0, \quad u = u(x,t)$$

The zero-curvature representation:

$$\vec{\Psi}_x(\lambda,x,t) = U(\lambda,x,t)\vec{\Psi}(\lambda,x,t), \quad \vec{\Psi}_t(\lambda,x,t) = V(\lambda,x,t)\vec{\Psi}(\lambda,x,t),$$

$$U = \begin{bmatrix} -i\lambda & iu(x,t) \\ \overline{iu(x,t)} & i\lambda \end{bmatrix},$$

$$V(\lambda,x,t) = \begin{bmatrix} -2i\lambda^2 + iu(x,t)\overline{u(x,t)} & 2i\lambda u(x,t) - u_x(x,t) \\ 2i\lambda\overline{u(x,t)} + \overline{u_x(x,t)} & 2i\lambda^2 - iu(x,t)\overline{u(x,t)} \end{bmatrix},$$

where

$$\vec{\Psi}(\lambda,x,t) = \begin{bmatrix} \Psi^1(\lambda,x,t) \\ \Psi^2(\lambda,x,t) \end{bmatrix}.$$

The auxiliary linear problem

Alternatively,

$$\mathcal{L}\vec{\Psi}(\lambda, x, t) = \lambda\vec{\Psi}(\lambda, x, t),$$

$$\mathcal{L} = \left[\begin{array}{c|c} i\partial_x & u(x,t) \\ \hline -u(x,t) & -i\partial_x \end{array} \right].$$

The operator $\mathcal{L}$ is not self-adjoint, and the spectrum of this problem typically contains complex points.
We consider the x-periodic problem:

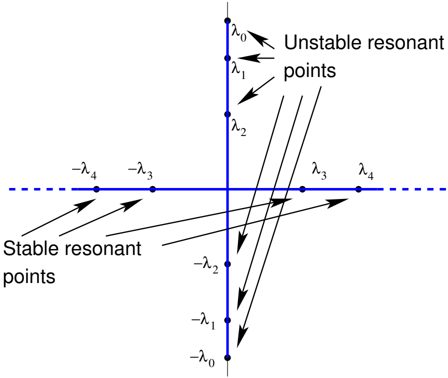
$$u(x+L, t) = u(x, t).$$

Let us recall the spectral data for the periodic problem.

The spectrum of unperturbed problem

The unperturbed spectral curve Γ_0:

$$\mu^2 = \lambda^2 + 1.$$



$$\mu_n = \frac{\pi n}{L}, \quad \lambda_n = \sqrt{\mu_n^2 - 1}, \quad \mathrm{Re}\,\lambda_n + \mathrm{Im}\,\lambda_n > 0, \quad n = 0, 1, 2, \ldots \infty.$$

The spectral data

In the periodic theory of the NLS equation the following two spectral problems are used to define the spectral data:

1. The spectral problem on the line, i.e. the spectral problem in $L^2(\mathbb{R})$. It is also called the **main spectrum**.
2. The spectral problem on the interval $[x_0, x_0 + L]$ with the following Dirichlet-type boundary conditions:

$$\Psi^1(\lambda, x_0, t) = \Psi^1(\lambda, x_0 + L, t) = 0.$$

This spectrum is called the **auxiliary spectrum** or **divisor**.

Remark. Many authors use the following symmetric boundary condition:

$$\Psi^1(\lambda, x_0, t) + \Psi^2(\lambda, x_0, t) = \Psi^1(\lambda, x_0 + L, t) + \Psi^2(\lambda, x_0 + L, t) = 0.$$

This approach has the following advantage: all divisor points are located in a compact area of the spectral curve, but it requires one extra divisor point and increases the complexity of the formulas.

To define the spectrum of the problem on the line, it is convenient to introduce the monodromy matrix. Consider the matrix equation

$$L\hat{\Psi}(\lambda, x, t_0) = \lambda\hat{\Psi}(\lambda, x, t_0),$$

where $\hat{\Psi}$ is a 2×2 matrix with the initial condition

$$\hat{\Psi}(\lambda, x_0, t_0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then the monodromy matrix $\hat{T}(\lambda, x_0, t_0)$ is defined by:

$$\hat{T}(\lambda, x_0, t_0) = \hat{\Psi}(\lambda, x_0 + L, t_0).$$

The eigenvalues and eigenvectors of $T(\lambda, x_0, t_0)$ are defined on a two-sheeted covering of the λ-plane. This Riemann surface Γ is called the **spectral curve**.

The spectral curve Γ is well-defined and does not depend on time.

The eigenvectors of $T(\lambda, x_0, t_0)$ are the Bloch eigenfunctions of L

The spectral curve

Let us denote:

$$\kappa(\gamma) = e^{iLp(\gamma)}.$$

The multivalued function $p(\gamma)$ is called **quasimomentum.** Its differential $dp(\gamma)$ is well-defined and meromorphic on Γ, all periods of dp are pure real.

The spectrum of L is exatly the projection of the set

$$\mathrm{Im}\, p(\gamma) = 0.$$

to the λ-plane.

For $\lambda \in \mathbb{R}$ the matrix U is skew-hermitian, and the monodromy matrix is unitary, therefore the whole real line lies in the spectrum of L in $L^2(\mathbb{R})$.

The end points of the spectrum are the branch points of Γ:

$$\kappa(\gamma) = \pm 1. \tag{1}$$

Equation (1) is also satisfied at the double points.

The spectral curve

All real double points in the SfNLS theory are removable, i.e. they do not arise in the inverse spectral transform. But a finite number of non-removable complex double points may be present.

Equivalently, the branch and double points of Γ are exactly the eigenvalues of L on the spaces of periodic and antiperiodic functions:

$$
\begin{aligned}
&L\vec{\Psi}(\gamma,x,t)=\lambda(\gamma)\vec{\Psi}(\gamma,x,t),\\
&\vec{\Psi}(\gamma,x+L,t)=\pm\vec{\Psi}(\gamma,x,t),\quad \gamma\in\Gamma.
\end{aligned}
$$

If

$$\vec{\Psi}(\lambda, x, t) = \begin{bmatrix} \Psi^1(\lambda, x, t) \\ \Psi^2(\lambda, x, t) \end{bmatrix}$$

solves the auxiliary linear problem

$$L\vec{\Psi}(\lambda, x, t) = \lambda\vec{\Psi}(\lambda, x, t),$$

then

$$\vec{\Psi}^{+}(\lambda, x, t) = \begin{bmatrix} \overline{\Psi^2(\lambda, x, t)} \\ -\overline{\Psi^2(\lambda, x, t)} \end{bmatrix}$$

satisfies the same equation with complex conjugate eigenvalue:

$$L\vec{\Psi}^{+}(\lambda, x, t) = \bar{\lambda}\vec{\Psi}^{+}(\lambda, x, t). \tag{2}$$

It immediately implies that Γ is real, i.e. the set of branch points of Γ is invariant with respect to the complex conjugation.

Finite-gap potentials

Potential $u(x,t)$ is called **finite-gap** if the spectral curve Γ is algebraic, i.e., if it can be written in the form

$$\nu^2 = \prod_{j=1}^{2g+2} (\lambda - E_j).$$

It means that Γ has only a finite number of branch points and non-removable double points. Such solutions can be written in terms of the Riemann theta-functions. Any smooth, periodic in x solution admits arbitrarily good finite gap approximation, for any fixed time interval

A good approximation: We open only gaps associated with the unstable modes. The perturbations associated with stable modes remain small for all times..

The auxiliary spectrum

The auxiliary spectrum **(the divisor)** is defined as the set of points $\gamma \in \Gamma$ such that the first component of the Bloch eigenfunction is equal to 0 at the point x_0, t_0.

$$\begin{aligned} L\vec{\Psi}(\gamma, x, t) &= \lambda(\gamma)\vec{\Psi}(\gamma, x, t), \\ \vec{\Psi}(\gamma, x + L, t)) &= \kappa(\gamma)\vec{\Psi}(\gamma, x, t), \\ \Psi^1(\gamma, x_0, t_0) &= 0. \end{aligned}$$

Equivalently, the auxiliary spectrum coincides with the set of zeroes of the first component of the Bloch eigenfunction:

$$\Psi^1(\gamma, x_0, t_0) = 0. \tag{3}$$

Spectral data for a small perturbation of the constant solution

Remark. Using the scaling $x \to \alpha^2 x$, $t \to \alpha t$ the generic case can be reduced to the case $u_0(x,t) = e^{2it}$.
Let $u(x) = 1 + \epsilon(x)$, where $|\epsilon(x)| \ll 1$, $\epsilon(x+L) = \epsilon(x)$. Then

$$U = \begin{bmatrix} -i\lambda & i(1+\epsilon(x)) \\ i(1+\bar{\epsilon}(x)) & i\lambda \end{bmatrix},$$

where

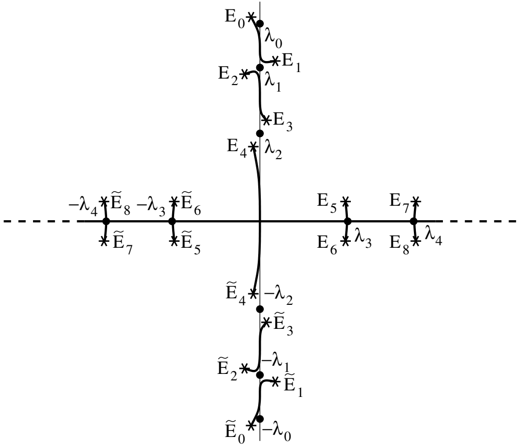
$$\epsilon(x,0) = \sum_{j\geq 1} \left(c_j e^{ik_j x} + c_{-j} e^{-ik_j x} \right), \quad k_j = \frac{2\pi}{L} j, \quad |c_j| = O(\epsilon),$$

It is convenient to write:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1, \quad \mathcal{L}_0 = \begin{bmatrix} i\partial_x & 1 \\ -1 & -i\partial_x \end{bmatrix}, \quad \mathcal{L}_1 = \begin{bmatrix} 0 & \epsilon(x) \\ -\overline{\epsilon(x)} & 0 \end{bmatrix},$$

and the spectral data for $\mathcal{L}$ will be calculated using the perturbation theory near the spectral data for $\mathcal{L}_0$.

Perturbed spectral curve



Perturbations of **real** resonant points generate **stable** perturbations of solutions. They can be neglected.
Perturbations of **imaginary** resonant points generate **exponentially growing** perturbations of solutions.

Perturbed spectral curve

$$u(x,0) = 1 + \epsilon(x,0), \quad \epsilon(x,0) = \sum c_j e^{i\frac{2\pi}{L}jx}, \quad c_j \ll 1, \quad c_0 = 0.$$

Let us define:

$$\alpha_j = \overline{c_j} - (\mu_j + \lambda_j)^2 c_{-j}, \quad \beta_j = \overline{c_{-j}} - (\mu_j - \lambda_j)^2 c_j,$$

$$\tilde{\alpha}_j = \overline{c_j} - (\mu_j - \lambda_j)^2 c_{-j}, \quad \tilde{\beta}_j = \overline{c_{-j}} - (\mu_j + \lambda_j)^2 c_j.$$

$$\mu_j = \frac{\pi j}{L}, \quad \lambda_j^{\pm} = \pm\sqrt{\mu_j^2 - 1}, \quad \mathrm{Re}\,\lambda_j^{+} + \mathrm{Im}\,\lambda_j^{+} > 0, \quad j = 1, 2, \ldots \infty.$$

Then:

$$E_l = \lambda_j \mp \frac{1}{2\lambda_j}\sqrt{\alpha_j \beta_j} + O(\epsilon^2), \quad l = 2j-1, 2j,$$

$$\lambda(\gamma_n) = \lambda_n + \frac{1}{4\lambda_n}[\alpha_n + \beta_n] + O(\epsilon^2), \quad p(\gamma_n) = \frac{1}{4\mu_n}[\alpha_n - \beta_n] + O(\epsilon^2)$$

Homoclinic orbit in this approximation: $\alpha_j \neq 0, \beta_j = 0$.

For the unstable modes it is convenient to define:

$$\phi_j = \arccos\left(\frac{\pi}{L} j\right) = \arccos\left(\frac{k_j}{2}\right).$$

Then

$$\lambda_j = i\sin(\phi_j), \ \ \mu_j = \cos(\phi_j), \ \ \sigma_j = k_j\sqrt{4 - k_j^2},$$

$$\alpha_j = \overline{c_j} - e^{2i\phi_j} c_{-j}, \ \ \beta_j = \overline{c_{-j}} - e^{-2i\phi_j} c_j,$$

where σ_j is the linear increment of the unstable mode.

The theta-functional solutions

A. R. Its., V. P. Kotljarov "Explicit formulas for solutions of the Nonlinear Schrdinger equation" (Ukrainian) Dokl. Ukrain. SSR, Ser. A, no. 11, (1976), 965–968.

$$u(x,t) = \frac{\theta(\vec{A}(\infty_-) - \vec{U}_1 x - \vec{U}_2 t - \vec{A}(D) - \vec{K})}{\theta(\vec{A}(\infty_+) - \vec{U}_1 x - \vec{U}_2 t - \vec{A}(D) - \vec{K})} \times \tag{4}$$

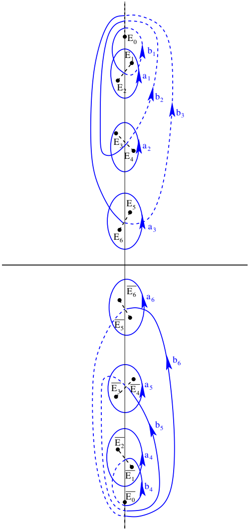
$$\times \frac{\theta(\vec{A}(\infty_+) - \vec{A}(D) - \vec{K})}{\theta(\vec{A}(\infty_-) - \vec{A}(D) - \vec{K})} \cdot u(0,0) \cdot \exp(2it)(1 + O(\epsilon^2)),$$

$$\theta(z|B) = \sum_{n_j} \exp\left[2\pi i \sum_j n_j z_j + \pi i \sum_{j,k} b_{jk} n_j n_k\right], \quad j,k = 1,\ldots,g.$$

Here $\vec{A}(\gamma)$ denotes the Abel transform, $\vec{K}$ is the vector of Riemann constants, B denotes the matrix of periods, $\vec{U}_1$, $\vec{U}_2$ are some periods of meromorphic differentials.

We need some explicit approximate formulas.

We use the following basis of cycles:



The Riemann period matrix

For the matrix of period we obtain (here $g = 2N$):

$$b_{jj} = \frac{1}{\pi i} \log \left[\frac{\sqrt{\alpha_j \beta_j}}{8\,\mathrm{Im}(\lambda_j)\,(\mu_j)^2} \right] + \ldots, \quad 1 \le j \le N,$$

$$\exp\left(\pi i b_{jj}\right) = \frac{\sqrt{\alpha_j \beta_j}}{8\,\mathrm{Im}(\lambda_j)\,(\mu_j)^2} + O(\epsilon^2), \quad 1 \le j \le N,$$

$$b_{j+N,j+N} = -\overline{b_{j,j}}, \quad 1 \le j \le N.$$

$$b_{kj} = \frac{1}{\pi i} \log \left[\frac{|\,\mathrm{Im}(\hat{\lambda}_j - \hat{\lambda}_k)|}{\hat{\lambda}_j \hat{\lambda}_k + \hat{\mu}_j \hat{\mu}_k + 1} \right] + O(\epsilon^2) \ \text{ for all } \ j \ne k.$$

$$\hat{\lambda}_j = \begin{cases} \lambda_j & \text{if } \ 1 \le j \le N \\ \overline{\lambda_{j-N}} & \text{if } \ N+1 \le j \le 2N \end{cases}, \quad \hat{\mu}_j = \begin{cases} \mu_j & \text{if } \ 1 \le j \le N \\ \mu_{j-N} & \text{if } \ N+1 \le j \le 2N \end{cases}$$

Final formulas

One can choose integration path so that

$$u(x,t) = \exp(2it)\frac{\theta(\vec{A}(\infty_-) - \vec{U}_1 x - \vec{U}_2 t - \vec{A}(D))}{\theta(-\vec{A}(\infty_-) - \vec{U}_1 x - \vec{U}_2 t - \vec{A}(D))}\times \tag{5}$$

$$\times\frac{\theta(-\vec{A}(\infty_-) - \vec{A}(D))}{\theta(\vec{A}(\infty_-) - \vec{A}(D))}\times u(0,0)\times(1+O(\epsilon^2)).$$

$$\vec{A}(\infty_-) = \begin{bmatrix} -\frac{1}{4} - \frac{\phi_1}{2\pi} \\ \vdots \\ -\frac{1}{4} - \frac{\phi_N}{2\pi} \\ \hline -\frac{1}{4} + \frac{\phi_1}{2\pi} \\ \vdots \\ -\frac{1}{4} + \frac{\phi_N}{2\pi} \end{bmatrix} + O(\epsilon^2), \quad \vec{A}(D) = \begin{bmatrix} \frac{1}{2\pi i}\log\left[\frac{\alpha_1}{\sqrt{\alpha_1\beta_1}}\right] \\ \vdots \\ \frac{1}{2\pi i}\log\left[\frac{\alpha_N}{\sqrt{\alpha_N\beta_N}}\right] \\ \hline \frac{1}{2\pi i}\log\left[-\frac{\tilde{\alpha}_1}{\sqrt{\tilde{\alpha}_1\tilde{\beta}_1}}\right] \\ \vdots \\ \frac{1}{2\pi i}\log\left[-\frac{\tilde{\alpha}_N}{\sqrt{\tilde{\alpha}_N\tilde{\beta}_N}}\right] \end{bmatrix} + \ldots,$$

Final formulas

$$\vec{U}_1 = \left[\begin{array}{c} -\frac{\mu_1}{\pi} \\ -\frac{\mu_2}{\pi} \\ \vdots \\ -\frac{\mu_N}{\pi} \\ \hline -\frac{\mu_1}{\pi} \\ -\frac{\mu_2}{\pi} \\ \vdots \\ -\frac{\mu_N}{\pi} \end{array}\right], \quad \vec{U}_2 = \left[\begin{array}{c} -\frac{2\lambda_1\mu_1}{\pi} \\ -\frac{2\lambda_2\mu_2}{\pi} \\ \vdots \\ -\frac{2\lambda_N\mu_N}{\pi} \\ \hline \frac{2\lambda_1\mu_1}{\pi} \\ \frac{2\lambda_2\mu_2}{\pi} \\ \vdots \\ \frac{2\lambda_N\mu_N}{\pi} \end{array}\right] + O(\epsilon^2),$$

One unstable mode

In the case of one unstable mode only, the simplest nontrivial initial condition excites just that mode:

$$u(x,0) = 1 + \epsilon\left(c_1 e^{ik_1 x} + c_{-1} e^{-ik_1 x}\right),$$

where c_1 and c_{-1} are arbitrary $O(1)$ complex parameters.
Problem: Calculate the time of the first rogue wave appearance and its position. Calculate the periodicity of appearances in terms of the Cauchy data.
Akhmediev 1-breather:

$$A_1(x,t;\phi_1,x_1,t_1,\rho) =$$

$$= e^{2it+i\rho}\frac{\cosh[\Sigma_1(t-t_1)+2i\phi_1]+\sin\phi_1\cos[K_1(x-x_1)]}{\cosh[\Sigma_1(t-t_1)]-\sin\phi_1\cos[K_1(x-x_1)]},$$

where

$$K_1 = 2\cos\phi_1, \quad \Sigma_1 = 2\sin(2\phi_1),$$

Approximate genus 2 solution

Approximation of the genus 2 solution at a finite time interval:

$$u(x,t) = \sum_{m=0}^{n} A_1\left(x,t;\phi_1,x_1^{(m)},t_1^{(m)},\rho^{(m)}\right) - \frac{1-e^{4in\phi_1}}{1-e^{4i\phi_1}} e^{2it}, \quad x \in [0,L],$$

where

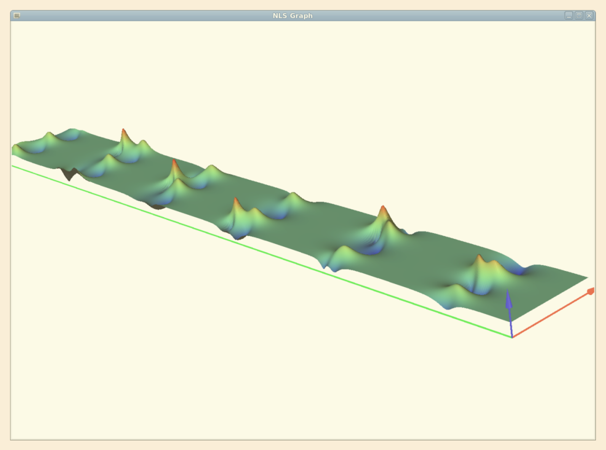
$$x_1^{(n)} = X_1^{+} + (n-1)\Delta_1^{(x)}, \;\; t_1^{(n)} = T_1(|\alpha_1|) + (n-1)T_p, \;\; \rho^{(n)} = 2\phi_1 + (n-1)4\phi_1,$$

$x_1^{(n)}$ denote the position of the maximum at the n-th appearance,
$t_1^{(n)}$ denotes the time of the n-th appearance,

$$X_1^{+} = \frac{\arg(\alpha_1) - \phi_1 + \pi/2}{k_1}, \quad \Delta_1^{(x)} = \frac{\arg(\alpha_1\beta_1)}{k_1}.$$

$$T_1(|\alpha_1|) = \frac{1}{\sigma_1}\log\left(\frac{\sigma_1^2}{2|\alpha_1|}\right), \quad T_p = \frac{2}{\sigma_1}\log\left(\frac{\sigma_1^2}{2\sqrt{|\alpha_1\beta_1|}}\right).$$

These formulas can be aslo derived directly using the matched asymptotic expansions.



Two unstable mode solution.