Matrix extensions of multidimensional dispersionless integrable systems and SDYM equations on the self-dual background

L.V. Bogdanov

L.D. Landau ITP RAS

Introduction

L V Bogdanov, SDYM equations on the self-dual background, J. Phys. A: Math. Theor. 50 19LT02 (2017)

L.V. Bogdanov and M.V. Pavlov, Linearly degenerate hierarchies of quasiclassical SDYM type, Journal of Mathematical Physics 58, 093505 (2017)

Multidimensional dispersionless integrable systems: Lax pairs of the type

$$[X_1, X_2] = 0,$$

$$X_1 = \partial_{t_1} + \sum_{i=1}^{N} F_i \partial_{x_i} + F_0 \partial_\lambda, \quad X_2 = \partial_{t_2} + \sum_{i=1}^{N} G_i \partial_{x_i} + G_0 \partial_\lambda.$$

λ - 'spectral parameter', functions F_k, G_k are holomorphic in λ and depend on the variables t_1, t_2, x_n. We will consider polynomials (or Laurent polynomials) in λ (meromorphic functions).

Dispersionless limits of integrable equations (dKP, dispersionless 2DTL hierarchy), Plebański heavenly equations, hyper-Kähler hierarchies belong to this class.

Matric extension - covariant vector fields of the form

$$\nabla_{X_1} = X_1 + A_1, \quad \nabla_{X_2} = X_2 + A_2,$$

A_1, A_2 are matrix functions of space-time variables holomorphic in λ (polynomials, Laurent polynomials, meromorphic functions).
Lax pairs of this structure were already present in Zakharov and Shabat (1979).
The commutator of two covariant vector fields contains vector field part and matrix (Lie algebraic) part,

$$[\nabla_{X_1}, \nabla_{X_2}] = [X_1, X_2] + X_1 A_2 - X_2 A_1 + [A_1, A_2]$$

Compatibility condition - vector fields (dispersionless equations)

$$[X_1, X_2] = 0$$

Compatibility condition - matrix part (matrix equations on the dispersionless background)

$$X_1 A_2 - X_2 A_1 + [A_1, A_2] = 0$$

Compatibility conditions imply (local) existence of common solutions for linear equations.

$$X_1\psi_i = 0, \quad X_2\psi_i = 0.$$

N independent solutions $\psi_i(\lambda, t_1, t_2, \mathbf{x})$ (Frobenius theorem) - 'wave fuctions' for dispersionless integrable system (background), general solution

$$\psi = f(\psi_1, \dots, \psi_N)$$

Wave function for extended system

$$\nabla_{X_1}\Phi_0 = (X_1 + A_1)\Phi_0 = 0, \quad \nabla_{X_2}\Phi_0 = (X_2 + A_2)\Phi_0 = 0,$$

locally Φ may be cosidered as series in λ. general solution

$$\Phi = \Phi_0 F(\psi_1, \dots, \psi_N),$$

where F is matrix complex analytic function.

It is easy to check that extended linear equations are equivalent to

$$(X_1\Phi)\Phi^{-1} = -A_1, \quad (X_2\Phi)\Phi^{-1} = -A_2$$

where A_1, A_2 are polynomials (Laurent polynomials, meromorphic functions). This is a characteristic analytic property of Φ, important for algebraic definition of the hierarchy (Lax-Sato equations) and for construction of solutions.

To construct Φ, we may consider matrix RH problem

$$\Phi_+ = \Phi_- R(\psi_1, \ldots, \psi_N),$$

defined on some oriented curve γ in the complex plane, or matrix $\bar\partial$ problem

$$\bar\partial\Phi = \Phi R(\psi_1, \ldots, \psi_N),$$

defined in some region G, and $\psi_i(\lambda, \mathbf{t})$ are arbitrary wave functions of dispersionless Lax pair

SDYM equations

SDYM (ASDYM) equations

$$\mathbf{F} = \pm * \mathbf{F}$$

represent SD (ASD) condition for the two-form (field intensity, connection curvature)

$$\mathbf{F} = \mathrm{d}\mathbf{A} + \mathbf{A} \wedge \mathbf{A}$$

the gauge field (potential) $\mathbf{A}$ is a one-form (connection form) taking its values in some Lie algebra (we will consider general matrix-valued form). Full Yang-Mills equation

$$D * \mathbf{F} = 0$$

SDYM are conformally invariant and depend only on conformal structure. SD (ASD) conformal structure $[g]$ (ASD or SD part of the Weyl tensor vanishes):

$$W = \pm * W,$$

for real case exist only for Riemannian (Euclidean) signature and neutral signature $(++--)$. Complexification allows to consider both cases on equal footing.

Integrable background geometries

Atiyah M.F., Hitchin N.J., Singer I.M., Self-duality in four-dimensional Riemannian geometry, Proc. Roy. Soc. London Ser. A 362 (1978), 425–461.

There is a curved twistor space as long as the conformal structure on 4-manifold is selfdual. SDYM equations for selfdual conformal structure are integrable by twistor approach.

David M.J. Calderbank, SIGMA 10 (2014), 034, 51 pages

SDYM equations (and their reductions) are integrable in some nonflat geometries described by dispersionless integrable equations.

We will go opposite direction, starting from dispersionless integrable equations and extending integrable structures (Lax pairs, dressing scheme, the hierarchy) for gauge field equations on the background.

Theorem (Dunajski, Ferapontov and Kruglikov (2014))

There exist local coordinates (z, w, x, y) such that any ASD conformal structure in signature $(2, 2)$ is locally represented by a metric

$$\frac{1}{2}g = dwdx - dzdy - F_y dw^2 - (F_x - G_y)dwdz + G_x dz^2,$$

where the functions $F,\ G : M^4 \to \mathbb{R}$ satisfy a coupled system of third-order PDEs,

$$\begin{aligned} &\partial_x(Q(F)) + \partial_y(Q(G)) = 0, \\ &(\partial_w + F_y\partial_x + G_y\partial_y)Q(G) + (\partial_z + F_x\partial_x + G_x\partial_y)Q(F) = 0, \end{aligned} \quad (1)$$

where

$$Q = \partial_w\partial_x - \partial_z\partial_y + F_y{\partial_x}^2 - G_x{\partial_y}^2 - (F_x - G_y)\partial_x\partial_y.$$

System (1) arises as $[X_1, X_2] = 0$ from the dispersionless Lax pair

$$X_1 = \partial_z - \lambda\partial_x + F_x\partial_x + G_x\partial_y + f_1\partial_\lambda,$$
$$X_2 = \partial_w - \lambda\partial_y + F_y\partial_x + G_y\partial_y + f_2\partial_\lambda.$$

Due to compatibility conditions, f_1 and f_2 can be expressed through F and G,

$$f_1 = -Q(G), \quad f_2 = Q(F),$$
$$Q = \partial_w\partial_x - \partial_z\partial_y + F_y{\partial_x}^2 - G_x{\partial_y}^2 - (F_x - G_y)\partial_x\partial_y.$$

Correspondence between ASD conformal structures and integrable system defined by generic commuting vector fields.
Real case with the signature (2,2) or, generally, complex analytic case may be considered.
Reductions:
Dunajski system - null Kähler case, divergence free vector fields
$f_1, f_2 = 0$ (no ∂_λ in the vector fields), divergence free - *Plebanski's second heavenly equation* (ASD, Ricci flat)

Integrability properties of this Lax pair

The hierarchy, Lax-Sato equations, the dressing scheme - Bogdanov, Dryuma and Manakov (2007)
The structure of the hierarchy in terms of vector fields

$$
\begin{aligned}
X_1^n &= \partial_{z^n} - \lambda^n \partial_x + F_1^n(\lambda)\partial_x + G_1^n(\lambda)\partial_y + f_1^n(\lambda)\partial_\lambda, \\
X_2^n &= \partial_{w^n} - \lambda^n \partial_y + F_2^n(\lambda)\partial_x + G_2^n(\lambda)\partial_y + f_2^n(\lambda)\partial_\lambda,
\end{aligned}
$$

where we have two infinite sets of times z^n, w^n and two 'basic' variables x, y, the coefficients of vector fields are polynomials in λ of the order $n-1$. Multidimensional version contains N infinite sets of times and N 'basic' variables.

Extension of the Lax pair

Consider a gauge field $\mathbf{A}$ in some (matrix) Lie algebra and 'covariant vector fields' X_1, X_2

$$\nabla_{X_1} = \partial_z - \lambda\partial_x + F_x\partial_x + G_x\partial_y + f_1\partial_\lambda + A_1,$$
$$\nabla_{X_2} = \partial_w - \lambda\partial_y + F_y\partial_x + G_y\partial_y + f_2\partial_\lambda + A_2$$

(here A_1, A_2 do not depend on λ). Lax pairs of this structure were already present in Zakharov and Shabat (1979).

The commutator of two covariant vector fields contains vector field part and Lie algebraic part,

$$[\nabla_{X_1}, \nabla_{X_2}] = [X_1, X_2] + X_1 A_2 - X_2 A_1 + [A_1, A_2]$$

Demanding both parts to be equal to zero, from the first part we get the system describing conformally ASD metric, and the second part gives the system for A_1, A_2

$$\partial_x A_2 = \partial_y A_1,$$
$$(\partial_z + F_x\partial_x + G_x\partial_y)A_2 - (\partial_w + F_y\partial_x + G_y\partial_y)A_1 + [A_1, A_2] = 0.$$

Standard ASDYM case (trivial plane background)

For $F = G = 0$ we have

$$X_1 = \partial_z - \lambda \partial_x,$$
$$X_2 = \partial_w - \lambda \partial_y,$$

$$\frac{1}{2} g = dwdx - dzdy.$$

The extended Lax pair takes the form

$$\nabla_{X_1} = \partial_z - \lambda \partial_x + A_1,$$
$$\nabla_{X_2} = \partial_w - \lambda \partial_y + A_2,$$

and the commutativity condition is

$$\partial_x A_2 = \partial_y A_1,$$
$$\partial_z A_2 - \partial_w A_1 + [A_1, A_2] = 0.$$

This a well known form of ASDYM equations for constant metric g in a special gauge.

General case

1. Geometry

Extended Lax pair gives a general form of ASDYM equations for arbitrary conformally ASD metric g in signature (2,2) (locally, up to transformations of coordinates and a gauge).

2. Integrability

Extended Lax pair belongs to the hierarchy which unites ASDYM hierarchy and generic 4-dimensional dispersionless hierarchy. Lax-Sato equations and dressing scheme can be constructed for this hierarchy.

Geometry

Given: conformally ASD metric g with signature (2,2) (ASD conformal structure) and ASD gauge field with a connection form $\mathbf{A}$. The corresponding gauge curvature form is $\mathbf{F} = \mathrm{d}\mathbf{A} + \mathbf{A} \wedge \mathbf{A}$, it satisfies the ASDYM equation

$$\mathbf{F} = - * \mathbf{F}$$

First step:
following Dunajski, Ferapontov and Kruglikov, we find local coordinates (z, w, x, y) such that ASD conformal structure is locally represented by a metric

$$\frac{1}{2}g = dwdx - dzdy - F_y dw^2 - (F_x - G_y)dwdz + G_x dz^2,$$

Second step:
notice that for this metric due to ASDYM equation we have

$$F_{34} = 0,$$

where we have used inverse matrix to metric g defined by symmetric bivector

$$\frac{1}{2}Q = \partial_w\partial_x - \partial_z\partial_y + F_y\,\partial_x^2 + (G_y - F_x)\,\partial_x\partial_y - G_x\,\partial_y^2$$

$\det g = \det Q = 1$ (for this metric $F^{12} = F_{34}$). Then it is possible to choose a gauge such that

$$A_3 = A_4 = 0,$$

and we have only two nontrivial gauge field components A_1, A_2.

Third step:
we will prove that ASDYM equations for A_1, A_2 for the metric g coincide with Lie algebraic part of compatibility equations for extended Lax pair.

Tetrad of one-forms

The conformal structure is represented by (DFK)

$$g = 2(\mathbf{e}^{00'}\mathbf{e}^{11'} - \mathbf{e}^{01'}\mathbf{e}^{10'}),$$

where the tetrad of one-forms is

$$\begin{aligned}
\mathbf{e}^{00'} &= dw, \\
\mathbf{e}^{10'} &= dz, \\
\mathbf{e}^{01'} &= dy - G_y dw - G_x dz, \\
\mathbf{e}^{11'} &= dx - F_y dw - F_x dz.
\end{aligned}$$

Tetrad of vector fields

The dual tetrad of vector fields is

$$
\begin{aligned}
\mathbf{e}_{00'} &= \partial_w + F_y\partial_x + G_y\partial_y, \quad (+A_2) \\
\mathbf{e}_{10'} &= \partial_z + F_x\partial_x + G_x\partial_y, \quad (+A_1) \\
\mathbf{e}_{01'} &= \partial_y, \\
\mathbf{e}_{11'} &= \partial_x,
\end{aligned}
$$

symmetric bivector reads

$$
Q = 2(\mathbf{e}_{00'}\mathbf{e}_{11'} - \mathbf{e}_{01'}\mathbf{e}_{10'}).
$$

ASDYM equations for this tetrad take the form

$$
F_{00'10'} = 0, \quad F_{00'11'} = F_{10'01'}
$$

For gauge field curvature $\mathbf{F}$ in the tetrad basis we use a standard formula

$$\mathbf{F}(\mathbf{u},\mathbf{v}) = \nabla_{\mathbf{u}}\nabla_{\mathbf{v}} - \nabla_{\mathbf{v}}\nabla_{\mathbf{u}} - \nabla_{[\mathbf{u},\mathbf{v}]}$$

for arbitrary vector fields $\mathbf{u}$, $\mathbf{v}$. Taking into account the structure of tetrade and the fact that for our gauge $A_3 = A_4 = 0$, we see that the third term doesn't contain a gauge field, and for the curvature components we get

$$\begin{aligned}
&F_{00'10'} = (\partial_w + F_y\partial_x + G_y\partial_y)A_1 - (\partial_z + F_x\partial_x + G_x\partial_y)A_2 - [A_1, A_2],\\
&F_{00'11'} = -\partial_x A_2, \quad F_{10'01'} = -\partial_y A_1
\end{aligned}$$

Thus ASDYM equations read

$$\begin{aligned}
&(\partial_w + F_y\partial_x + G_y\partial_y)A_1 - (\partial_z + F_x\partial_x + G_x\partial_y)A_2 - [A_1, A_2] = 0,\\
&\partial_x A_2 = \partial_y A_1,
\end{aligned}$$

which coincides with the Lie algebraic part of commutativity condition for extended vector fields Lax pair.

Gauge-invariant SDYM equations

Lax pair

$$\nabla_{X_1} = \partial_z + F_x\partial_x + G_x\partial_y + A_1 - \lambda(\partial_x + B_1) + f_1\partial_\lambda,$$
$$\nabla_{X_2} = \partial_w + F_y\partial_x + G_y\partial_y + A_2 - \lambda(\partial_y + B_2) + f_2\partial_\lambda$$

Compatibility condition (matrix part)

$$\begin{aligned}
&\partial_x B_2 - \partial_y B_1 + [B_1, B_2] = 0,\\
&\partial_x A_2 - (\partial_w + F_y\partial_x + G_y\partial_y)B_1 - [B_1, A_2]\\
&\qquad = \partial_y A_1 - (\partial_z + F_x\partial_x + G_x\partial_y)B_2 - [B_2, A_1],\\
&(\partial_z + F_x\partial_x + G_x\partial_y)A_2 - (\partial_w + F_y\partial_x + G_y\partial_y)A_1\\
&\qquad +[A_1, A_2] + f_2 B_1 - f_1 B_2 = 0.
\end{aligned}$$

Represent ASDYM equations for the ASD conformal structure

$$\frac{1}{2}g = dwdx - dzdy - F_y dw^2 - (F_x - G_y)dwdz + G_x dz^2,$$

First heavenly type extended Lax pair

Motivated by the Kähler case, we suggest to consider conformal structure defined by the symmetric bivector

$$\tfrac{1}{2}q = a\,\partial_w \cdot \partial_{\tilde{w}} + b\,\partial_z \cdot \partial_{\tilde{w}} + c\,\partial_w \cdot \partial_{\tilde{z}} + d\,\partial_z \cdot \partial_{\tilde{z}},$$

and corresponding extended Lax pair

$$\begin{aligned}
\nabla_{X_1} &= \partial_{\tilde{z}} - \lambda(a\partial_w + b\partial_z) + (\lambda^2 f_1 + \lambda g_1)\partial_\lambda + A_1 - \lambda B_1,\\
\nabla_{X_2} &= \partial_{\tilde{w}} + \lambda(c\partial_w + d\partial_z) + (\lambda^2 f_2 + \lambda g_2)\partial_\lambda + A_2 + \lambda B_2,
\end{aligned}$$

Vector fields part of commutation relations gives seven equations for eight functions because of conformal freedom. To fix representative of conformal structure and close the system of equations, it is convenient to use the condition $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$, in this case three independent coefficients of bivector satisfy three second-order equations, and the conformal structure depends on 6 arbitrary functions of three variables.

Einstein-Weyl geometry, Manakov-Santini system and monopole equation

An important (2+1)-dimensional example is provided by the Manakov-Santini system. Extended Lax pair

$$\nabla_{X_1} = \partial_y - (\lambda - v_x)\partial_x + u_x\partial_\lambda + A,$$
$$\nabla_{X_2} = \partial_t - (\lambda^2 - v_x\lambda + u - v_y)\partial_x + (u_x\lambda + u_y)\partial_\lambda + \lambda A + B,$$

A, B are gauge field components. Vector field part of commutation relations gives the Manakov-Santini system

$$\begin{aligned} u_{xt} &= u_{yy} + (uu_x)_x + v_x u_{xy} - u_{xx} v_y, \\ v_{xt} &= v_{yy} + u v_{xx} + v_x v_{xy} - v_{xx} v_y, \end{aligned} \tag{2}$$

describing general Einstein-Weyl geometry (DFK2014) (Lorentzian signature for the real case), and matrix part of compatibility conditions read

$$A_y - B_x = 0,$$
$$(\partial_y + v_x\partial_x)B - (\partial_t + (v_y - u)\partial_x)A + u_x A + [A, B] = 0$$

For the potential Φ, $A = \Phi_t$, $B = \Phi_y$ we have

$$\Phi_{tx} - \Phi_{yy} - [\Phi_x, \Phi_y] - \partial_x(u\Phi_x) + v_y\Phi_{xx} - v_x\Phi_{xy} = 0,$$

where u, v satisfy Manakov-Santini system describing Einstein-Weyl geometry.
This system represents a general local form of monopole equations on Einstein-Weyl background (up to coordinate transformations and a gauge).

Matrix dressing on the geometric background

Generally, we may consider matrix RH problem

$$\Phi_+ = \Phi_- R(\psi_1, \psi_2, \psi_3),$$

defined on some oriented curve γ in the complex plane, or matrix $\bar\partial$ problem

$$\bar\partial\Phi = \Phi R(\psi_1, \psi_2, \psi_3),$$

defined in some region G, and $\psi_i(\lambda, \mathbf{t})$ are arbitrary wave functions of dispersionless Lax pair

$$\begin{aligned}
X_1\psi_i &= (\partial_z - \lambda\partial_x + F_x\partial_x + G_x\partial_y + f_1\partial_\lambda)\psi_i = 0,\\
X_2\psi_i &= (\partial_w - \lambda\partial_y + F_y\partial_x + G_y\partial_y + f_2\partial_\lambda)\psi_i = 0,
\end{aligned}$$

defined on γ or in G.

Let us suggest the existence of solution Φ of RH (or $\bar{\partial}$) problem having no zeroes and normalized by 1 at infinity. Then $X_1\Phi$, $X_2\Phi$ satisfy the same problem ($[X_1, R] = [X_2, R] = 0$), and the functions

$$(X_1\Phi)\Phi^{-1},\ (X_2\Phi)\Phi^{-1}$$

are holomorphic in the complex plane.
Considering the behaviour at infinity, we get

$$(X_1\Phi)\Phi^{-1} = \partial_x\Phi_1(\mathbf{t}),$$
$$(X_2\Phi)\Phi^{-1} = \partial_y\Phi_1(\mathbf{t}),$$

or the solution for the extended Lax pair with the gauge field

$$A_1 = \partial_x\Phi_1(\mathbf{t}),\ A_2 = \partial_y\Phi_1(\mathbf{t}).$$

Dropping the normalization condition at infinity, we will get solution for gauge-invariant extended Lax pair.

There are important reductions connected with existence of *polynomial wave functions* for dispersionless Lax pair $\psi = P^n(\lambda)$, coefficients of the polynomial depends on times. A class of special ASDYM solutions for these geometries is defined by the problems

$$\Phi_+ = \Phi_- R(P^n) \quad \text{or}$$
$$\bar{\partial}\Phi = \Phi R(P^n).$$

Another important reduction of geometry: *linearly-degenerate case* (no ∂_λ in dispersionless Lax pair, λ is one of the wave functions),

$$\Phi_+ = \Phi_- R(\lambda, \psi_1, \psi_2) \quad \text{or}$$
$$\bar{\partial}\Phi = \Phi R(\lambda, \bar{\lambda}, \psi_1, \psi_2).$$

In this case ASDYM Lax pair admits rational (in λ) solutions with simple stationary poles (correspond to δ-functions in the $\bar{\partial}$ problem), which can be calculated explicitly.

From the dressing scheme to the hierarchy

1. **Dressing for vector fields.** Nonlinear vector Riemann-Hilbert problem (e.g. on the unit circle, here we don't discuss the question of reductions)

$$\begin{aligned}
\Psi^0_{\text{in}} &= F_0(\Psi^0_{\text{out}}, \Psi^1_{\text{out}}, \Psi^2_{\text{out}}),\\
\Psi^1_{\text{in}} &= F_1(\Psi^0_{\text{out}}, \Psi^1_{\text{out}}, \Psi^2_{\text{out}}),\\
\Psi^2_{\text{in}} &= F_2(\Psi^0_{\text{out}}, \Psi^1_{\text{out}}, \Psi^2_{\text{out}}),
\end{aligned}$$

the expansions at infinity are

$$\begin{aligned}
\Psi^0_{\text{out}} &= \lambda + \sum_{n=1}^{\infty} \Psi^0_n(\mathbf{t}^1, \mathbf{t}^2)\lambda^{-n},\\
\Psi^1_{\text{out}} &= \sum_{n=0}^{\infty} t^1_n (\Psi^0)^n + \sum_{n=1}^{\infty} \Psi^1_n(\mathbf{t}^1, \mathbf{t}^2)\lambda^{-n}\\
\Psi^2_{\text{out}} &= \sum_{n=0}^{\infty} t^2_n (\Psi^0)^n + \sum_{n=1}^{\infty} \Psi^2_n(\mathbf{t}^1, \mathbf{t}^2)\lambda^{-n},
\end{aligned}$$

inside the unit circle the functions are analytic.

Ψ^0, Ψ^1, Ψ^2 will give the wave fuctions for the hierarchy of commuting vector fields, defined through coefficients of expansion of these functions.

2. **Matrix dressing on the background.** Consider a matrix Riemann-Hilbert problem

$$\Phi_{\text{in}} = \Phi_{\text{out}} R(\Psi^0_{\text{out}}, \Psi^1_{\text{out}}, \Psi^2_{\text{out}}),$$

Φ is normalized by 1 at infinity and analytic inside and outside the unit circle,

$$\Phi_{\text{out}} = 1 + \sum_{n=1}^{\infty} \Phi_n(\mathbf{t}^1, \mathbf{t}^2)\lambda^{-n}$$

Expansions of Ψ, Φ give coefficients for extended Lax pair, Φ is a wave function. A general wave function is given by the expression $\Phi F(\Psi^0, \Psi^1, \Psi^2)$, F is arbitrary matrix function.
For constant metric g corresponding to trivial vector fields we have

$$\Psi^0 = \lambda, \quad \Psi^1 = x + \lambda z, \quad \Psi^2 = y + \lambda w,$$

and we get standard Riemann-Hilbert problem for ASDYM.

The vector fields part of the dressing scheme implies analyticity in the complex plane of the form (no dicontinuity on the unit circle)

$$\omega = \left| \frac{D(\Psi^0, \Psi^1, \Psi^2)}{D(\lambda, x_1, x_2)} \right|^{-1} \mathrm{d}\Psi^0 \wedge \mathrm{d}\Psi^1 \wedge \mathrm{d}\Psi^2,$$

where $x_1 = t_0^1$, $x_2 = t_0^2$ are lowest times of the hierarchy, and from matrix Riemann problem we get analyticity of the matrix-valued form

$$\Omega = \omega \wedge \mathrm{d}\Phi \cdot \Phi^{-1}.$$

Analyticity of these forms imply the relations

$$(\omega_{\text{out}})_- = \left(\left| \frac{D(\Psi^0_{\text{out}}, \Psi^1_{\text{out}}, \Psi^2_{\text{out}})}{D(\lambda, x_1, x_2)} \right|^{-1} \mathrm{d}\Psi^0_{\text{out}} \wedge \mathrm{d}\Psi^1_{\text{out}} \wedge \mathrm{d}\Psi^2_{\text{out}} \right)_- = 0,$$

$$(\Omega_{\text{out}})_- = (\omega_{\text{out}} \wedge \mathrm{d}\Phi_{\text{out}} \cdot \Phi^{-1}_{\text{out}})_- = 0$$

for the series Ψ^0_{out}, Ψ^1_{out}, Ψ^2_{out}, Φ_{out}. These relations are generating relations for the hierarchy in terms of formal series, they are equivalent to the complete set of Lax-Sato equations of the hierarchy.
First relation gives Lax-Sato equations for the hierarchy of commuting polynomial in λ vector fields (here we drop subscript 'out' for the series):

$$\partial^k_n \Psi = \sum_{i=0}^{2} \left(\left(\frac{D(\Psi^0, \Psi^1, \Psi^2)}{D(\lambda, x_1, x_2)} \right)^{-1}_{ik} (\Psi^0)^n \right)_+ \partial_i \Psi,$$

where $1 \leqslant n < \infty$, $k = 1, 2$, $\partial_0 = \partial_\lambda$, $\partial_1 = \partial_{x_1}$, $\partial_2 = \partial_{x_2}$,
$\Psi = (\Psi^0, \Psi^1, \Psi^2)$.

The second generating relation gives Lax-Sato equations for Φ on the vector field background in terms of extended polynomial vector fields,

$$\partial_n^k \Psi = V_n^k(\lambda)\Psi,$$
$$\partial_n^k \Phi = \left(V_n^k(\lambda) - ((V_n^k(\lambda)\Phi)\cdot\Phi^{-1})_+\right)\Phi$$

First flows give exactly the extended Lax pair for ASDYM equations on ASD background, if we identify $z = t_1^1$, $w = t_1^2$, $x = x_1$, $y = x_2$.

Questions

- Solutions!
- Higher-dimensional case – what is the geometry?
- Lower-dimensional cases and reductions – known integrable systems on the background?

THANK YOU!