О соотношении динамики автогенератора с запаздывающей обратной связью и кольца связанных автогенераторов

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Chimera states in networks

Kuramoto and Battogtohk, 2002

\[ \frac{\partial \phi}{\partial t} = \omega - \int_{-\pi}^{\pi} G(x - x') \sin[\phi(x, t) - \phi(x', t) + \alpha] dx' \]

Network \(\Rightarrow\) coherent and incoherent domains

Instant phases

Average frequencies
(Virtual) chimera states

\[ \varepsilon x' = -\delta y - x + \beta f[x(s - 1)] \]
\[ y' = x \]

Coherent and incoherent phases

Space-time representation of a delayed dynamical system

\[ \dot{x}(t) = F(x(t), x(t - \tau)) \]

\[ t = \sigma + \theta T \]

\[ \sigma \in [0, T] \quad \text{-(pseudo) space} \]

\[ \theta \in \mathcal{N} \quad \text{-(slow) time} \]
(Virtual) chimera states

\[ \varepsilon x' = -\delta y - x + \beta f[x(s - 1)], \quad y' = x \]

Coexistence of “coherent” and “incoherent” domains

Reservoir computing

Larger et al, Opt. Express 2012

Complex network of nonlinear nodes

Optoelectronic element + delayed feedback
Reservoir computing

Larger et al, Opt. Express 2012
Single oscillator vs. ring of oscillators

Single oscillator with delayed feedback

\[ \frac{dx(t)}{dt} = f(x(t), x(t - \tau)) \]

Ring of oscillators with feed-forward delayed coupling

\[ \frac{dx_j(t)}{dt} = f(x_j(t), x_{j-1}(t - \sigma)) \]
Rings (loops) in neuroscience

Wang, TRENDS in Neurosciences 2001

Reverberation as potential mechanism for working memory

Thalamocortical loop (and others)

Local cortical recurrent loops
Ring motifs in complex networks

Sums of delays along fundamental semicycles matter:

**Reduction of delays**

\[ \tau_1 = \tau_1 + \tau_8 + \tau_6 + \tau_4 + \tau_3 + \tau_2 \]
\[ \tau_2 = \tau_1 + \tau_7 + \tau_4 + \tau_3 + \tau_2 \]
\[ \tau_3 = \tau_1 + \tau_8 + \tau_5 + \tau_2 \]

**Chaotic synchronization**

Lücken et al, EPL 2013

Kanter et al, EPL 2011
Single oscillator vs. ring of oscillators

Single oscillator with delayed feedback

\[ \frac{dx(t)}{dt} = f(x(t), x(t - \tau)) \]

Ring of oscillators with feed-forward delayed coupling

\[ \frac{dx_n(t)}{dt} = f(x_n(t), x_{n-1}(t - \sigma)) \]
Single oscillator vs. ring of oscillators

\[ \frac{dx(t)}{dt} = f(x(t), x(t - \tau)) \] \hspace{1cm} (1) = SINGLE

\[ \frac{dx_n(t)}{dt} = f(x_n(t), x_{n-1}(t - \sigma)) \] \hspace{1cm} (2) = RING

\( x(t) = h(t) \) is a \( T \)-periodic solution of (1) for \( \tau = \tau_0 \)

then \( x(t) = h(t) \) also solves (1) for \( \tau = \tau_0 + kT \)

moreover, \( x_n(t) = h(t + n\theta) \) is a solution of (2) for

\( \theta = MT/N \) and \( \sigma = \tau_0 + kT - \theta \), where \( M = 0, 1, \ldots, N-1 \) is wave-number
Example: Van-der-Pol oscillator

\[
\frac{d^2 x(t)}{dt^2} = \alpha[1 - x(t)^2] \frac{dx(t)}{dt} - x(t) + \kappa x(t - \tau)
\]

Single oscillator

Periodic solution

Ring of oscillators

Periodic solutions of single oscillator \(\Rightarrow\) Rotating waves in rings
Example: Van-der-Pol oscillator

\[
\frac{d^2 x(t)}{dt^2} = \alpha [1 - x(t)^2] \frac{dx(t)}{dt} - x(t) + \kappa x(t - \tau)
\]

**Single oscillator**

\[\tau_0 \Rightarrow \tau = \tau_0 + kT\]

**Ring of oscillators**

\[\tau_0 \Rightarrow \sigma = \tau_0 + (k - M/N)T\]
Stability analysis

Single oscillator

\[ \frac{d\delta(t)}{dt} = A(t)\delta(t) + B(t)\delta(t - \tau) \]

\[ A(t) = \partial_1 f[h(t), h(t - \tau)] \]
\[ B(t) = \partial_2 f[h(t), h(t - \tau)] \]

\[ \delta(t) = p(t)e^{\lambda t} \quad - \text{Floquet ansatz} \]

\[ \frac{dp(t)}{dt} = [A(t) - \lambda \text{Id}]p(t) + e^{-\lambda \tau} B(t)p(t - \tau) \]
Stability analysis

Ring of oscillators

\[
\frac{d\delta_n(t)}{dt} = \partial_1 f(h(t + n\theta), h(t + (n - 1)\theta - \sigma))\delta_n(t) \\
+ \partial_2 f(h(t + n\theta), h(t + (n - 1)\theta - \sigma))\delta_{n-1}(t - \sigma)
\]

\[
\delta_n(t) = r_n(t)e^{\lambda t} - \text{Floquet ansatz}
\]

\[
\frac{dr_n(t)}{dt} = [A(t + n\theta) - \lambda \text{Id}]r_n(t) + e^{-\lambda \sigma} B(t + n\theta)r_{n-1}(t - \sigma)
\]

\[
p_n(t) = r_n(t - n\theta) - \text{time shift}
\]

\[
p_n(t) = \sum_{m=1}^{N} \hat{p}_m(t)e^{2\pi imn/N} - \text{Fourier transformation}
\]

\[
\frac{d\hat{p}_m(t)}{dt} = [A(t) - \lambda_m \text{Id}]\hat{p}_m(t) + e^{-\lambda_m \sigma - i\psi_m} B(t)\hat{p}_m(t - \tau)
\]
Stability analysis

Ring of oscillators

\[
\frac{d \hat{p}_m(t)}{dt} = [A(t) - \lambda_m \mathbf{I}] \hat{p}_m(t) + e^{-\lambda_m \sigma - i \psi_m} B(t) \hat{p}_m(t - \tau)
\]

\[
\tau = \sigma + \theta
\]

Single oscillator

\[
\frac{dp(t)}{dt} = [A(t) - \lambda \mathbf{I}] p(t) + e^{-\lambda \tau} B(t) p(t - \tau)
\]

Characteristic eq.

\[
F(\lambda, e^{-\lambda \tau}) = 0
\]

Yanchuk and Perlikowski, Phys. Rev. E 2009
Spectrum for large delays

Characteristic eq.

Single oscillator

\[ F(\lambda, e^{-\lambda \tau}) = 0 \]

“Weak” spectrum

\[ \lambda = i \omega + \frac{\gamma}{\tau} \]

\[ \omega = \omega(\gamma) \]
Spectrum for large delays

Yanchuk and Perlikowski, Phys. Rev. E 2009

Characteristic eq.

**Single oscillator**

\[ F(\lambda, e^{-\lambda \tau}) = 0 \]

**Ring of oscillators**

\[ F(\lambda, e^{-\lambda \sigma - i \psi_m}) = 0 \]

The same “weak” spectrum

* Stability of periodic solutions of single oscillator and rotating waves in ring (with all wave-numbers) is the same for large delays
Ring with instant coupling

\[ \tau \Rightarrow \sigma = \tau - \frac{M}{NT} \]

\[ \sigma = 0 : \tau = \frac{M}{NT} \] - delay and period are resonant

Bifurcation diagram for single oscillator

waves in ring with instant coupling
Stability analysis

Characteristic eq.

**Single oscillator**

\[ F(\lambda, e^{-\lambda\tau}) = 0 \]

**Ring of oscillators with instant coupling**

\[ F(\lambda, e^{-i\psi_m}) = 0 \]

* Stability of periodic solutions of single oscillator for large delays is sufficient for stability of rotating waves in ring with instant coupling

* For large number of oscillators in ring \( N >> 1 \) it is also a necessary condition
Example: multi-jittering

Single oscillator with pulse delayed feedback

\[ \frac{d\varphi}{dt} = 1 + Z(\varphi) \sum_{t_p} \delta(t - t_p - \tau) \]

“Jittering” regimes with distinct inter-spike intervals
Multi-jittering waves

Bifurcation diagram for single oscillator

- Regular regimes
- Jittering regimes
Jittering rotating waves

Bifurcation diagram for single oscillator

- regular regimes
- jittering regimes

regular waves

jittering waves
Jittering rotating waves

Regular rotating waves

Jittering rotating waves

spikes

time

ISI

spikes

time
Conclusions

* Periodic solutions of single oscillator $\Rightarrow$ Rotating waves in rings

$$\tau_0 \Rightarrow \sigma = \tau_0 + (k - \frac{M}{N}) T$$

* Stability of periodic solutions of single oscillator
and rotating waves in ring (with all wave-numbers)
is the same for large delays

* Stability of periodic solutions of single oscillator for large delays
is sufficient for stability of rotating waves in ring with instant coupling

* For large number of oscillators in ring $N >> 1$
it is also a necessary condition

Klinshov, V., Shchapin, D., Yanchuk, S., Wolfrum, M., D’Huys, O., Nekorkin, V.
Embedding the dynamics of a single delay system into a feed-forward ring.
Eckhaus instability


\[ z' = (\alpha + i\beta)z - z|z|^2 + z_{\tau} \]

Multiplicity of coexisting periodic attractors

(a) \hspace{2cm} (b)
Complex patterns


\[ \dot{y} = \mu y - (1 + i \beta |y|^2)y + \eta y_d \]