Collapse vs. blow up in vortex stretching by the generalized Constantin-Lax-Majda equation

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3D Euler's equations of incompressible fluid motion

$$\frac{D\mathbf{u}}{Dt} := \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p, \quad \rho \equiv 1,$$

$$\mathrm{div}\,\mathbf{u} = 0$$

3D Euler equations in vorticity formulation

$$\partial_t \boldsymbol{\omega} + \mathbf{u}\cdot\nabla\boldsymbol{\omega} = \boldsymbol{\omega}\cdot\nabla\mathbf{u}, \quad \mathbf{x}\in\mathbb{R}^3,$$

$$\mathbf{u} = \nabla\times(-\Delta)^{-1}\boldsymbol{\omega}.$$

Vortex stretching term

$\boldsymbol{\omega}$ - vorticity

$\mathbf{u}$ - velocity of fluid

$\Rightarrow$ Biot-Savart law

$$\mathbf{u}(\mathbf{x},t) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{(\mathbf{x}-\mathbf{y})\times\boldsymbol{\omega}(\mathbf{y},t)}{|\mathbf{x}-\mathbf{y}|^3}\,d\mathbf{y}$$

Blow up in axisymmetric Euler equation in stream function-vorticity formulation[1]

$$\mathbf{u} = u^r(r,z,t)e_r + u^\theta(r,z,t)e_\theta + u^z(r,z,t)e_z$$

$$\partial_t u^\theta + u^r\partial_r u^\theta + u^z\partial_z u^\theta = -\frac{u^r u^\theta}{r},$$

$$\partial_t \omega^\theta + u^r\partial_r \omega^\theta + u^z\partial_z \omega^\theta = \partial_z\left(\frac{(u^\theta)^2}{r}\right) + \frac{u^r\omega^\theta}{r},$$

$$-\left(\Delta_x - \frac{1}{r^2}\right)\psi^\theta = \omega^\theta,$$

$$u^r = -\partial_z\psi^\theta, \quad u^z = \frac{1}{r}\partial_r(r\psi^\theta).$$

New variables $u_1 = \frac{u^\theta}{r}, \quad \omega_1 = \frac{\omega^\theta}{r}, \quad \psi_1 = \frac{\psi^\theta}{r}$

$$\Rightarrow \quad \begin{aligned} &\partial_t u_1 \boxed{+ u^r\partial_r u_1 + u^z\partial_z u_1} = 2\partial_z\psi_1 u_1, \\ &\partial_t\omega_1 \boxed{+ u^r\partial_r\omega_1 + u^z\partial_z\omega_1} = \partial_z\left((u_1)^2\right) \\ &-\left(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2\right)\psi_1 = \omega_1, \end{aligned}$$

If dropping these advection-type terms $\Rightarrow$ Blow up in u^θ

Another examples of blow up in approximate reductions of Euler equation[2-3]

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[1]T. Hou and Z. Lei et al, Commun. Pure Appl. Math. 62, 501 (2009).
[2]E.A. Kuznetsov, J. Fluid. Mech, 600, 167 (2008).
[3]D.S. Agafontsev, E.A. Kuznetsov, A.A. Mailybaev, IOP Conf. Ser.: Earth Environ. Sci., 231, 012002 (2019).

3D incompressible Euler equation in vorticity formulation

$$\partial_t \boldsymbol{\omega} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \mathbf{u}, \quad \mathbf{x} \in \mathbb{R}^3,$$

$$\mathbf{u} = \nabla \times (-\Delta)^{-1} \boldsymbol{\omega}.$$

Vortex stretching term

$\boldsymbol{\omega}$ - vorticity

$\mathbf{u}$ - velocity of fluid

$\Rightarrow$ Biot-Savart law

$$\mathbf{u}(\mathbf{x}, t) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(\mathbf{x} - \mathbf{y}) \times \boldsymbol{\omega}(\mathbf{y}, t)}{|\mathbf{x} - \mathbf{y}|^3} \, d\mathbf{y}$$

1D analogs: replace $\mathbf{u} = \nabla \times (-\Delta)^{-1} \boldsymbol{\omega}$ by the Hilbert transform[1,2]

$\omega_t = \omega u_x, \quad \omega, x \in \mathbb{R},$ - Constantin-Lax-Majda Eq.[2]

$$u_x = -\hat{H}\omega, \quad \hat{H}\omega(x) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{+\infty} \frac{\omega(x')}{x' - x} \mathrm{d}x'$$

[1]P.K. Choi, T. Hou et al, Commun. Pure Appl. Math. 70, 2218 (2017). G. Luo and T. Y. Hou, Proc. Natl. Acad. Sci. USA, 111 (2014), pp.12968–12973.

[2]P. Constantin, P. D. Lax, and A. Majda, Commun. Pure Appl. Math. 38, 715 (1985).

Generalization of Constantin-Lax-Majda equation to include advection

Focus on generalized Constantin-Lax-Majda Eq.[1,2], also related to surface quasi-geostrophic Eq. [3]

$$\boxed{\begin{aligned} &\omega_t + au\omega_x = \omega u_x \\ &u_x = -\hat{H}\omega, \end{aligned}}$$

a - advection strength

[1]S. De Gregorio, J. Stat. Phys. 59, 1251 (1990).
[2]H. Okamoto, T. Sakajo, and M. Wunsch, Nonlinearity 21, 2447 (2008).
[3]T. M. Elgindi and I.-J. Jeong, Archive for Rational Mechanics and Analysis 235, 1763 (2020)

Blow up in Constantin-Lax-Majda equation [1]

Expansion into function analytic in upper and lower complex half-planes:

$$\omega = \omega^+ + \omega^- \quad \Rightarrow \quad \hat{H}\omega = \mathrm{i}(\omega^+ - \omega^-)$$

If no advection *a=0* $\Rightarrow$ complete decoupling $\omega_t^+ = -\mathrm{i}(\omega^+)^2, \quad \omega_t^- = \mathrm{i}(\omega^-)^2$

$$\Rightarrow \qquad \omega^+(x,t) = \frac{\omega_0^+(x)}{1 + \mathrm{i}t\omega_0^+(x)} \quad \text{and} \quad \omega^-(x,t) = \frac{\omega_0^-(x)}{1 - \mathrm{i}t\omega_0^-(x)}$$

- finite-time singularity (blow up)[1]

$$\omega = \omega^+ + \omega^- = \frac{4\omega_0(x)}{[2 + t\hat{H}\omega_0(x)]^2 + t^2\omega_0^2(x)}$$

[1] P. Constantin, P. D. Lax, and A. Majda, Commun. Pure Appl. Math. 38, 715 (1985).

Assuming $\omega_0(x_0) = 0$ and expanding near $x = x_0$ and
$t_c := 2/\sup\{\mathcal{H}\omega_0(x) | \omega_0(x) = 0\}$

$$\Rightarrow \qquad \omega(x,t) = \frac{1}{t_c - t} \frac{4\xi\omega_0'(x_0)[\hat{H}\omega_0(x_0)]^2}{\Big([\mathcal{H}\omega_0(x_0)]^2 + 2\xi\hat{H}\omega_0'(x_0)\Big)^2 + 4\xi^2[\omega_0'(x_0)]^2} + O((t_c - t)^0)$$

$$\xi := \frac{x - x_0}{t_c - t}$$ - self-similar variable

In neglecting $O((t_c - t)^0)$ $\Rightarrow$ exact self-similar solution, i.e.
collapse-type blow up

$$\omega(x,t) = \frac{\mathrm{i}}{t_c - t}\left(\frac{\xi_+}{\xi - \xi_+} - \frac{\xi_-}{\xi - \xi_-}\right),$$

$$\xi_\pm = \frac{[\mathcal{H}\omega_0(x_0)]^2}{2[\mathcal{H}\omega_0'(x_0) \pm \mathrm{i}\omega_0'(x_0)]}$$

If $a \ll 1$ - still collapse-type blow up [1]

If $a = 1$ - qualitatively different type of blow up (expanding blow up) [2]

New results[3]: self-similar structure of singularities and blow up for all a

We into the complex plane of x. Contrary to the free surface dynamics, $\omega(x)$ is analytic in the strip about the real line with the complex Singularities appearing in complex conjugated pairs.

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[1] T. M. Elgindi and I.-J. Jeong, Archive for Rational Mechanics and Analysis 235, 1763 (2020).

[2] J. Chen, T. Y. Hou, and D. Huang, arXiv:1905.06387 (2019).

[3] P.M. Lushnikov, D.A. Silantyev, M. Siegel, Submitted to J. Nonlin. Sci. (2020), ArXiv:2010.01201.

Three main analytical results

1. Leading order singularity is of power law type:

$$\omega(x,t) = \frac{\omega_{-\gamma}(t)}{[x - \mathrm{i}v_c(t)]^{\gamma}} + \frac{\bar{\omega}_{-\gamma}(t)}{[x + \mathrm{i}v_c(t)]^{\gamma}}$$

$$\boxed{\begin{aligned} &\omega_t + au\omega_x = \omega u_x \\ &u_x = -\hat{H}\omega, \end{aligned}}$$

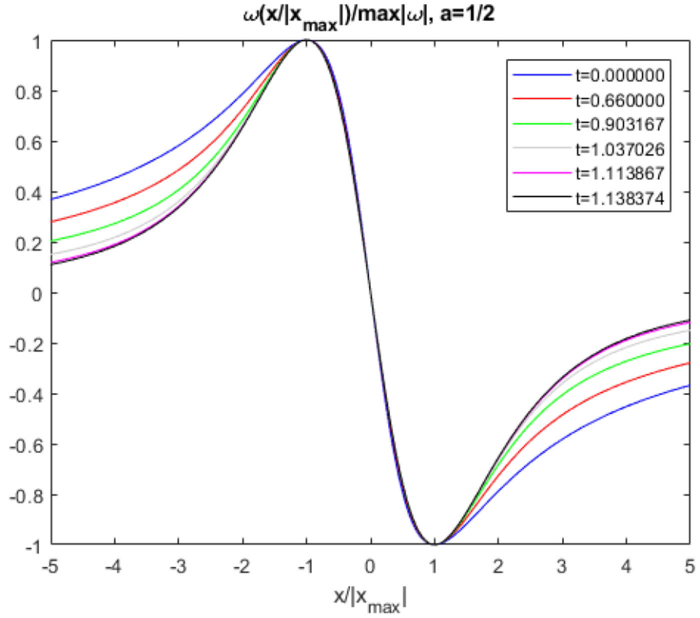
Balance of the most singular terms

$$\Rightarrow \qquad \boxed{\gamma = \frac{1}{1-a}}$$

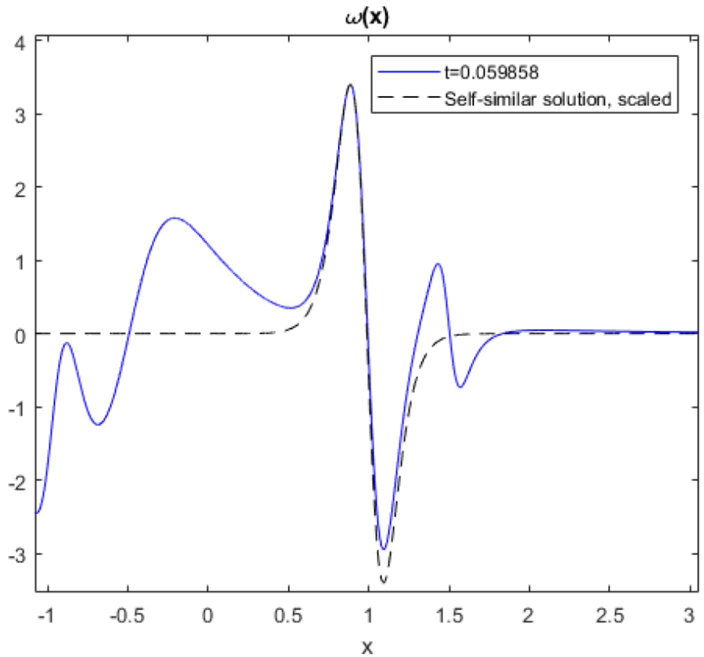
2. New exact collapse solution at *a=1/2* with a finite value of kinetic energy:

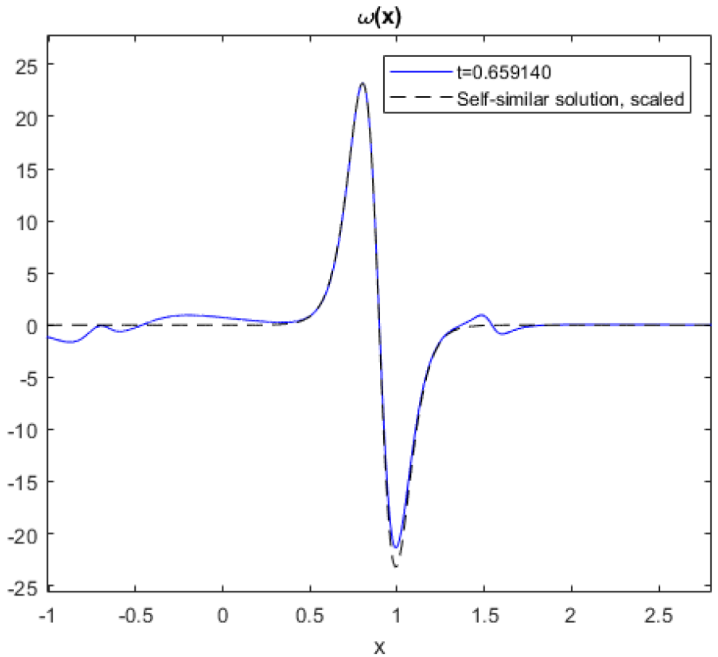
$$\omega(x,t) = \frac{1}{t_c - t}\frac{4\mathrm{i}\tilde{v}_c^2}{3}\left(\frac{1}{[\xi - \mathrm{i}\tilde{v}_c]^2} - \frac{1}{[\xi + \mathrm{i}\tilde{v}_c]^2}\right) \qquad \xi := \frac{x - x_0}{(t_c - t)^{1/3}}$$

Convergence of time-dependent solution to self-similar solution



Convergence of time-dependent solution to self-similar solution locally for gneric initial conditions

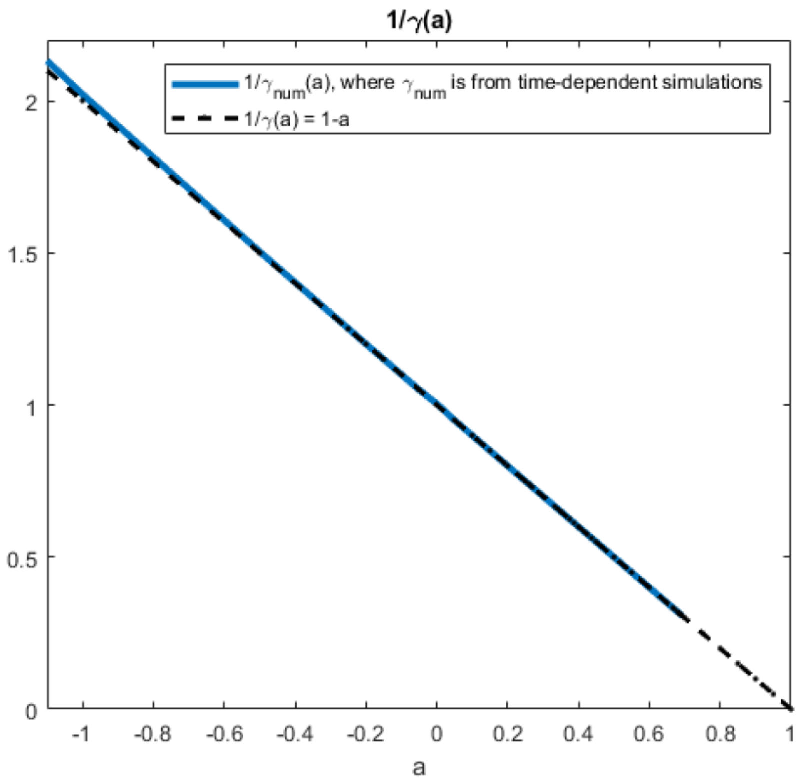




3. Proof that for $a \neq 0, 1/2$ with the solution has a nontrivial Laurent series starting from the most singular term with $\gamma = \dfrac{1}{1-a}$

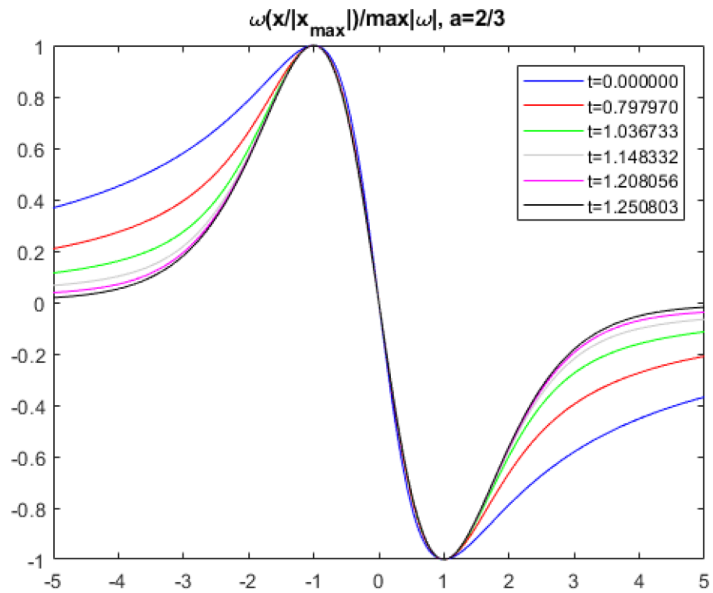
For example for $a = \frac{2}{3}$ with $\gamma = 3$,

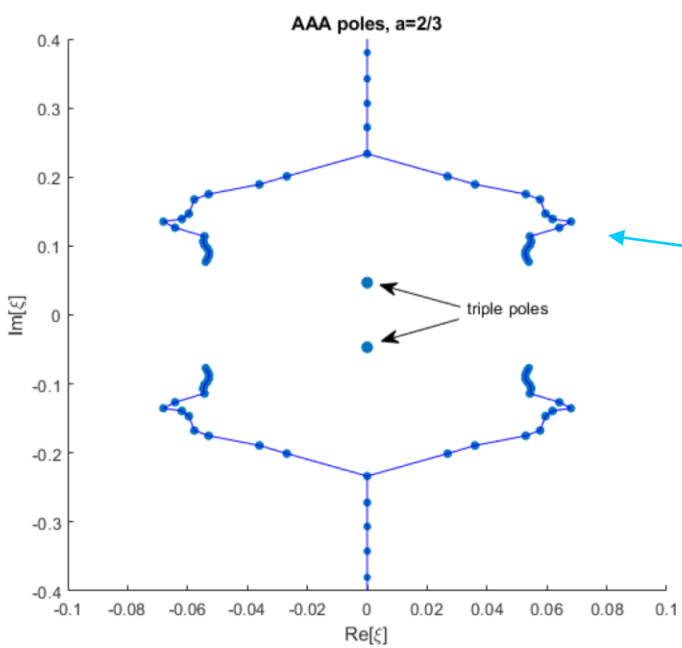
$$\omega(x,t) = \sum_{j=-3}^{\infty} -\mathrm{i}\tilde{\tilde{\omega}}_j(t) e^{\mathrm{i}\pi j/2}[x - \mathrm{i}v_c(t)]^j$$



$\gamma(a)$ from simulations vs. $\gamma = \dfrac{1}{1-a}$

Convergence of time-dependent solution to self-similar solution





Singularities in the complex plane

3. Self-similar blow up solutions for general a:

$$\omega = \frac{1}{\tau} f\left(\xi\right), \ \xi = \frac{x}{\tau^{\alpha}}, \ \tau = t_c - t$$

$\Longrightarrow$ Nonlinear eigenvalue problem for $\alpha(a)$:

$$f + \alpha \xi f_{\xi} = -a g f_{\xi} + f g_{\xi}, \quad g = \partial_{\xi}^{-1} \mathcal{H} f$$

Invariance to the stretching of self-similar coordinate : $\xi \to A\xi, \ A = const \in \mathbb{R}$

Nonlinear eigenvalue problem is solved by ensuring the convergence of the generalized Petviashvili method[1]

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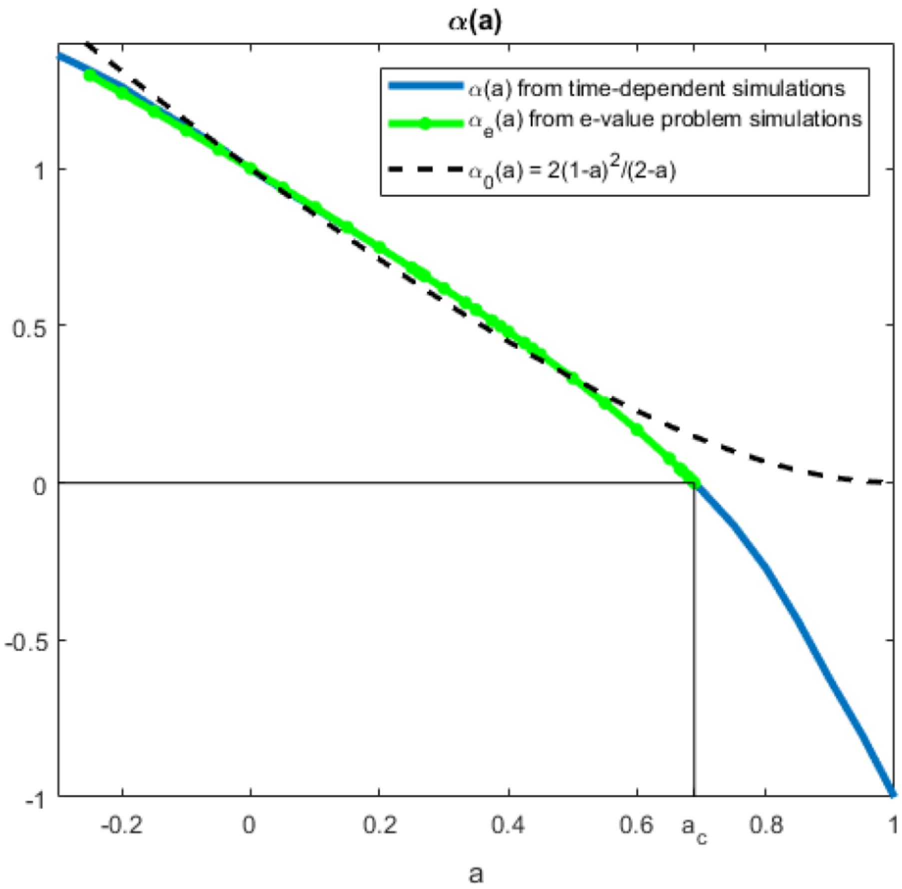
[1] Lakoba, T.I., Yang, J.: J. Comput. Phys. 226, 1668–1692 (2007)

Dependence $\alpha(a)$ and the threshold of the collapse from generalized Petviashvili method and direct time-dependent simulations



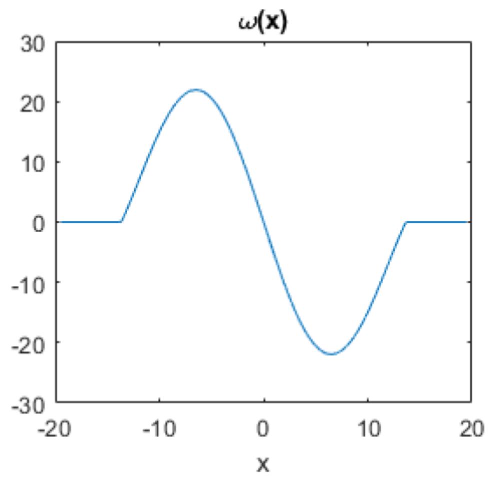
$$\omega = \frac{1}{\tau} f(\xi), \; \xi = \frac{x}{\tau^{\alpha}}, \; \tau = t_c - t.$$

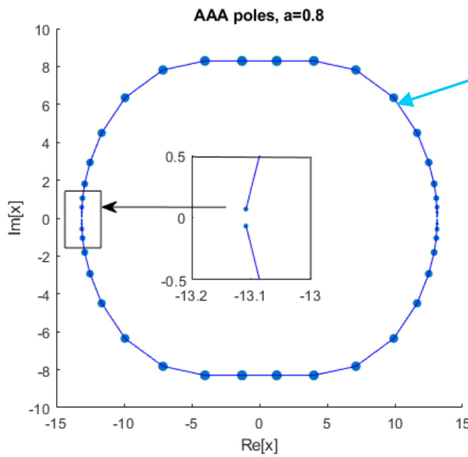
Critical value: $a_c = 0.6890665337007457\ldots$

$a < a_c$: collapse

$a > a_c$: infinitely fast expanding blow up solution with the finite support

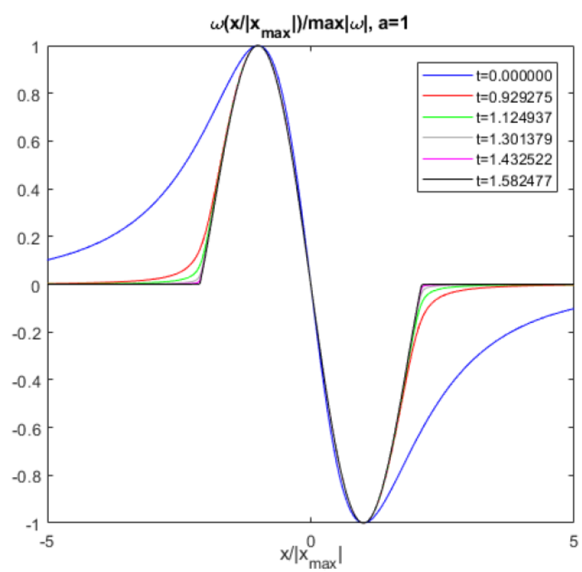
$a > a_c$: infinitely fast expanding blow up solution with the finite support





Singularities in the complex plane

Convergence of time-dependent solution to self-similar solution with the finite support



For $a = 1$ agrees with the results of Ref. [1]

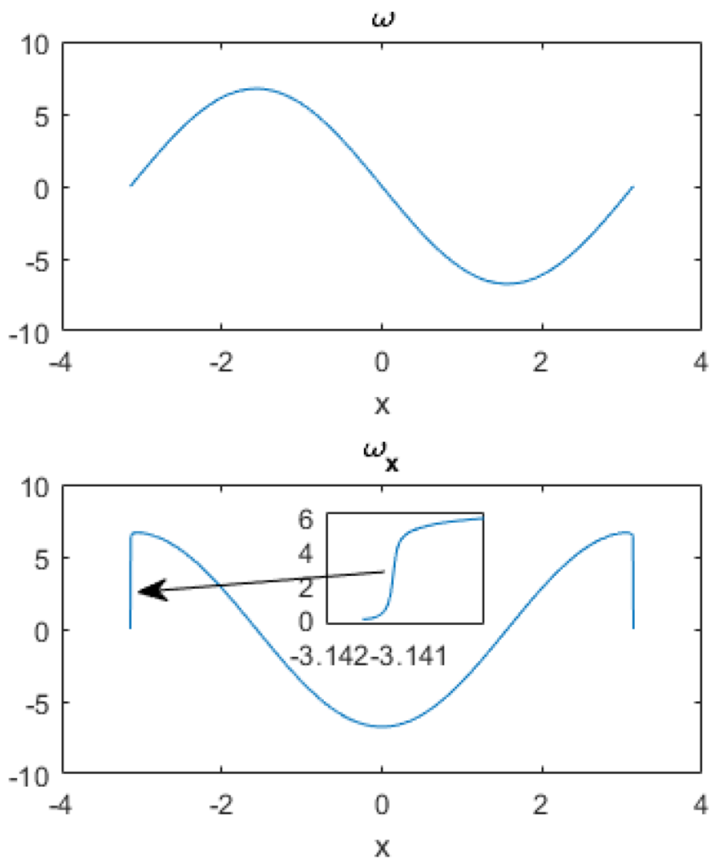
[1] J. Chen, T. Y. Hou, and D. Huang, arXiv:1905.06387 (2019).

Periodic boundary conditions

$a < a_c$: collapse as for the decaying boundary conditions.
$a > a_c$: expanding solution settles for large enough time at boundaries resulting
In periodic solution with the jump in the higher order derivatives at the boundary

Self-similar solution ("pure" blow up):

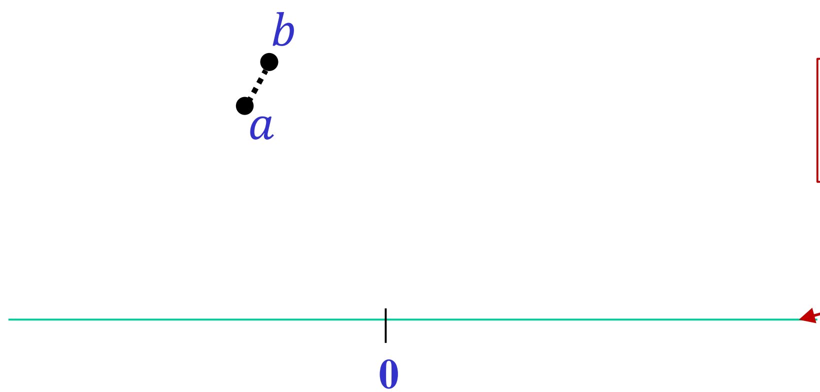
$$\omega(x,t) = \frac{1}{t_c - t} f(x)$$



Because ω is real-valued, singularities appear in complex conjugated pairs. Then using the splitting $\omega = \omega^{+} + \omega^{-}$ one can split each of these pairs.

Short branch cut approximation

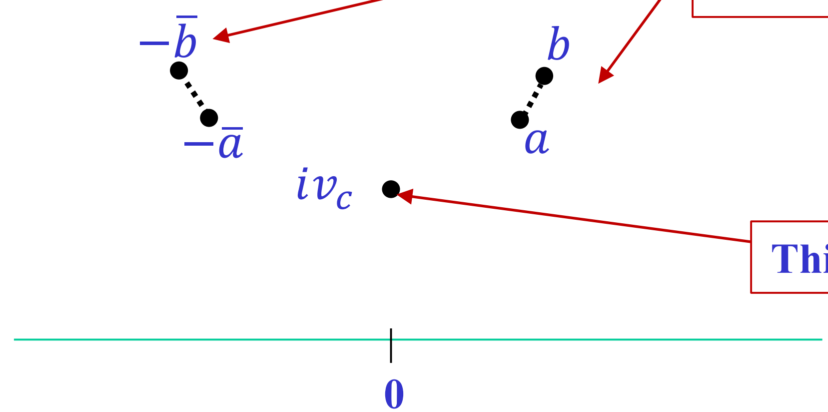
$$|a - b| \ll |Im(a)|$$



$\boldsymbol{x = x_r + i\, x_i}$
complex plane

Real line of x

Location of singularities of self-similar solution for *a=2/3* which implies $\boldsymbol{\gamma = 3}$



Two pairs of branch points

Third order pole

Short branch cut approximation

$$\omega^{-}(w,t) = \int_a^b \frac{\tilde{\omega}(w',t)\mathrm{d}w'}{w-w'}$$

$$\boxed{|a-b| \ll |Im(a)|}$$

$$\Rightarrow$$

Partial differential equation for short branch cut approximation

$$(\omega^{-})_t = 2a\mathrm{i}\omega_c\omega^{-} - 2a\mathrm{i}qq_c - \mathrm{i}aq(\omega^{-})_x + \mathrm{i}(\omega^{-})^2,$$

$$q_x = \omega^{-},$$

$$\omega_c(t) \equiv \omega^{+}(w_0(t),t), \quad q_c(t) \equiv \bar{q}(w_0(t),t), \quad w_0 = (a+b)/2$$

Conclusion and future directions

- Structure of singularities, self-similar solution and collapse threshold for the generalized Constantin-Lax-Majda equation.

-Advection prevents collapse above the threshold.

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[1]P.M. Lushnikov, D.A. Silantyev, M. Siegel, Collapse vs. blow up and global existence in the generalized Constantin-Lax-Majda equation,Submitted to J. Nonlin. Sci. (2020), ArXiv:2010.01201.