Russian Science Foundation

Динамика компактонов в сублинейном уравнении Кортевега – де Вриза

Д.Е. Пелиновский [1,2], А.В. Кокорина [2,3],

Алексей Слюняев [2-4], Е.Н. Пелиновский [2,3,4]

1) McMaster University, Hamilton, Canada

2) Институт прикладной физики РАН

3) Нижегородский государственный технический университет

4) НИУ Высшая школа экономики, Н. Новгород

Compactons

Compactons were originally introduced by P. Rosenau & J.M. Hyman in 1993 within the framework of the degenerate KdV equation with nonlinear advection and nonlinear dispersion,

$$u_t + \left(u^m\right)_x + \left(u^n\right)_{xxx} = 0$$

($m > 1$, $n > 1$), $u(x,t) \in \Re$, as solitons with compact support.

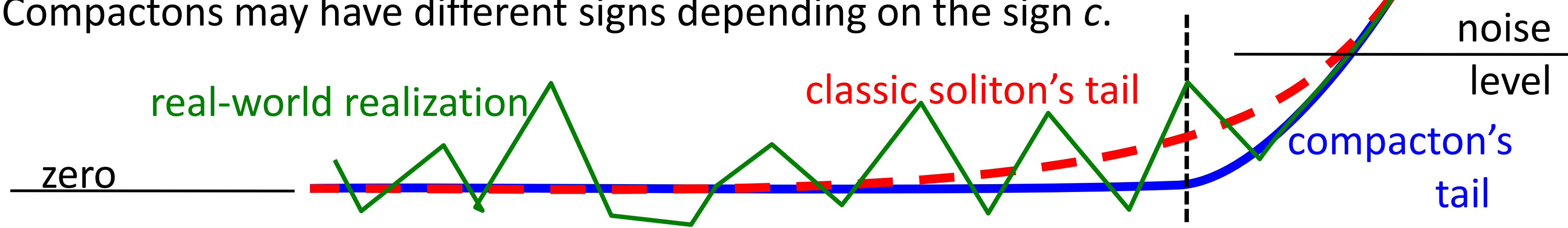
Equations of a similar form, arise in application to continuous limit of anharmonic oscillators in the elasticity theory, the magma dynamics, the surface waves on vorticity discontinuities, sedimentation.

Consider the case $n = 2$, $m = 2$, $u_t + \left(u^2\right)_x + \left(u^2\right)_{xxx} = 0$

The nonlinear ODE on the stationary wave solution $u(x,t) = u(\xi)$, $\xi = x-ct$ reads $u(u-c)+\left(u^2\right)_{\xi\xi} = 0$, its solution us not unique. A linkage of these two solution yields a localized solution in the form

$$u(x,t) = \begin{cases} \dfrac{4c}{3}\cos^2\dfrac{x-ct}{4} & if \quad |x-ct| \le 2\pi \\ 0 & otherwise \end{cases}$$

Compactons may have different signs depending on the sign c.



New compacton solutions

Compacton-type solutions may be found in the generalized KdV equation **with linear dispersion** (n = 1) and arbitrary power of nonlinearity $\alpha \in (0,1)$, ($u(x,t) \in \Re$)

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left[u\,|\,u\,|^{\alpha-1}\right] + \frac{\partial^3 u}{\partial x^3} = 0$$

As the nonlinear part of the equation is smaller than the linear approximation, this equation corresponds to a sub-linear system. This equation was introduced by P. Rosenau in 2007 and ‘rediscovered’ in our group by Efim Pelinovsky.
This equation appears in the theory of granular chains near the harmonic limit and in modeling of chemical reactions.

Alternatively, the equation may be written in the form
$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left[\left|u\right|^{\alpha} \operatorname{sgn} u\right] + \frac{\partial u}{\partial x^3} = 0$$

Assuming $u \sim x^{\beta}$, where β is a real constant, the derivative $\partial_x u^{\alpha+\beta}$ is not singular at $u = 0$ when $\beta \geq 1-\alpha$.

When the term of nonlinear advection is omitted, the equation reduces to the linear KdV equation $u_t + u_{xxx} = 0$, which may be exactly solved using the Fourier transform. Its dispersion relation $\omega = -k^3$ results in negative group velocities, $d\omega/dk = -3k^2 \leq 0$.

The ‘nonlinear velocity’ C_{nl}, provided by the nonlinear advection equation $u_t + C_{nl}u_x = 0$, $C_{nl} = -\,\alpha|\,u\,|^{\alpha-1}$, is always negative, similar to the defocusing modified KdV equation (there $C_{nl} = -6u^2$).

New compacton solutions

The sublinear KdV equation admits 3 conservation laws similar to classic mechanical systems,

$$M = \int_{-\infty}^{\infty} u dx \qquad P = \int_{-\infty}^{\infty} u^2 dx \qquad H = \int_{-\infty}^{\infty} \left(\frac{1}{2} u_x^2 + \frac{1}{1+\alpha} |u|^{1+\alpha} \right) dx$$

The compacton solutions have the form

$$u(x,t) = \begin{cases} \pm A \sin^{\frac{2}{1-\alpha}} \dfrac{\pi(x+ct)}{\lambda} & if \quad 0 \le x+ct \le \lambda \\ 0 & otherwise \end{cases}$$

where $\lambda = \dfrac{2\pi}{(1-\alpha)\sqrt{c}}$ and $A = \left[\dfrac{1}{(1+\alpha)c} \right]^{\frac{1}{1-\alpha}}$

The compactons may have either sign. They propagate to the left, $c > 0$.
Faster compactons are larger in amplitude and wider.

The term $\partial_x(u|u|^{\alpha-1})$ is always nonsingular ($\beta = 2/(1-\alpha)$).

The third derivative $u_{xxx} \sim x^{(3\alpha-1)/(1-\alpha)}$, therefore the compactons are solutions of the PDE in the classic sense when $1/3 < \alpha < 1$.

Compactons are energetically stable with respect to compact perturbations with the same support in the range of parameters $1/5 < \alpha < 1$.

If $2/(1-\alpha)$ is integer (e.g. $\alpha = 1/2$), then the nonlinear periodic wave $u(x) = \pm A[\sin(\pi x/\lambda)]^{2/(1-\alpha)}$ consists of finitely many Fourier harmonics, hence it may be efficiently analyzed using the Fourier series representation.

New compacton solutions

Numerical simulation of a particular example

The power $\alpha = 3/4$ was chosen to study the solutions of the sublinear KdV equation numerically,

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left[u|u|^{-\frac{1}{4}}\right] + \frac{\partial u}{\partial x^3} = 0 \quad \text{or} \quad \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left[|u|^{\frac{3}{4}} \operatorname{sgn} u\right] + \frac{\partial u}{\partial x^3} = 0$$

The equation admits 3 conservation laws, dM/dt=0, dP/dt=0, dE/dt=0, which were controlled in the simulations.

$$M = \int_{-\infty}^{\infty} u\,dx \qquad P = \int_{-\infty}^{\infty} u^2\,dx \qquad E = \int_{-\infty}^{\infty} \left(\frac{1}{2}u_x^2 + \frac{4}{7}|u|^{\frac{7}{4}}\right)dx$$

The compacton solution reads

$$u(x,t) = \begin{cases} A\sin^8 \dfrac{\pi(x+ct)}{\lambda} & \text{if} \quad 0 \le x+ct \le \lambda \\ 0 & \text{otherwise} \end{cases}$$

$$\lambda = \sqrt{56}\pi|A|^{1/2}$$

$$c = \frac{7}{8}|A|^{1/4}$$

The compactons may have either sign, $A > 0$ or $A < 0$. They propagate to the left, $c > 0$. Faster compactons are larger in amplitude and wider.

Two numerical codes were employed:

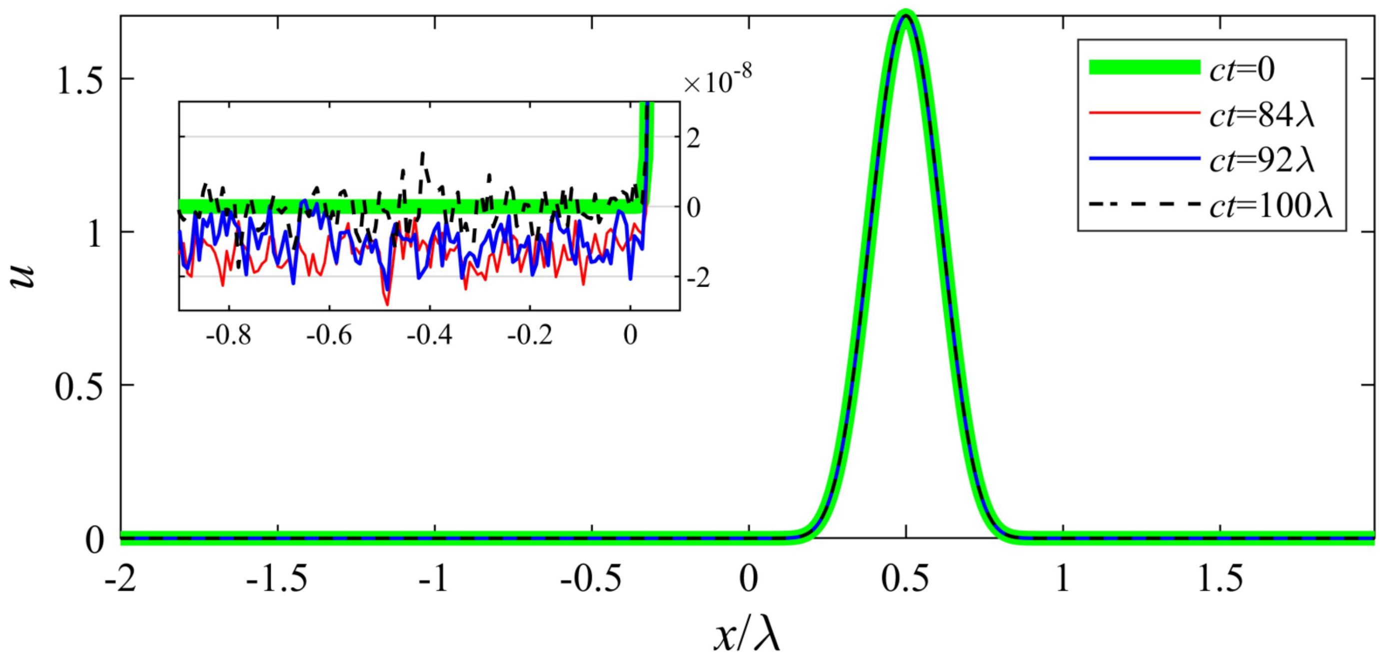
1 (main) Explicit split-step pseudo-spectral method with integration in time using the 4-order Runge-Kutta method.

2 (faster but less accurate) Implicit three-layer pseudo-spectral method.

New compacton solutions

Propagation of a single compacton

The initial condition is specified in the form of a single compacton with $c = 1$ ($A \approx 1.7$, $\lambda \approx 25$) in the periodic simulation domain of the total length $L = 4\lambda$.



The solution reproduces itself at the time instants $ct = nL$, where n is an integer.

Weak noisy oscillations arise in the course of the simulation, though they do not seem to grow in time.

The integrals of the equation are conserved with the relative error smaller than $1 \cdot 10^{-9}$.

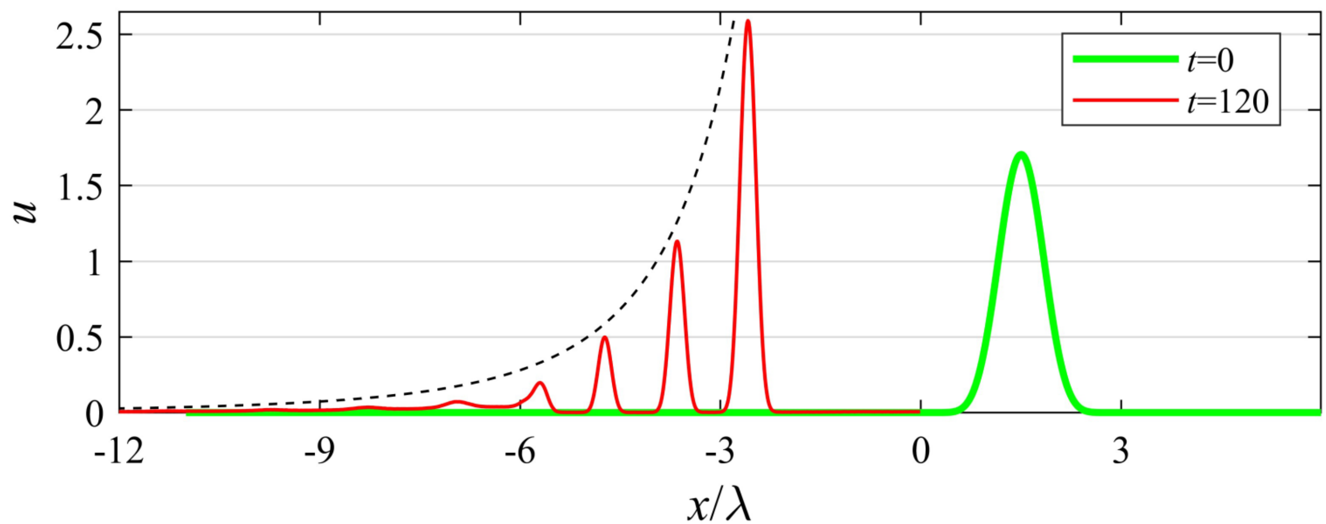
Hence, the numerical simulation confirms the stability of compactons with respect to small perturbations.

New compacton solutions

The Cauchy problem for pulse-like disturbances

The initial condition is specified in the form of a compacton with $A \approx 1.7$, but the shape is stretched (a wide initial condition) or squeezed (a narrow initial condition).

In the figures below the initial condition is thrice broader than a compacton of the same amplitude. The horizontal axis is scaled with the width λ of the parent compacton.



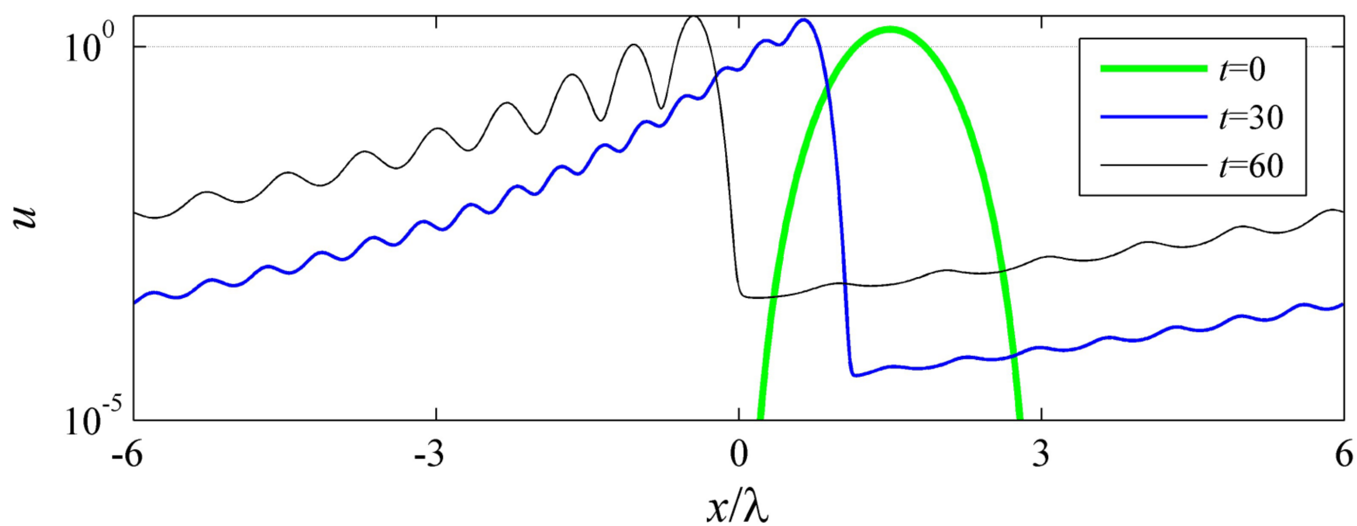
The initial condition evolves into a series of localized bell-shaped pulses, which propagate to the left according to the relation between the amplitude and velocity of compactons (see the dashed line). The compactons' tops are located to the right from the dashed line, what reveals the backward nonlinear shift of coordinates (to the right)

New compacton solutions

The Cauchy problem for a broad pulse-like disturbance

The generation of compactons at the initial stage of the evolution is better seen in semilogarythmic axes.

The broad initial condition produces a series of descending compactons. They are as many in number, as it is allowed by the size of the simulation domain. The compactons propagate to the left and later on will cause multiple mutual interactions.



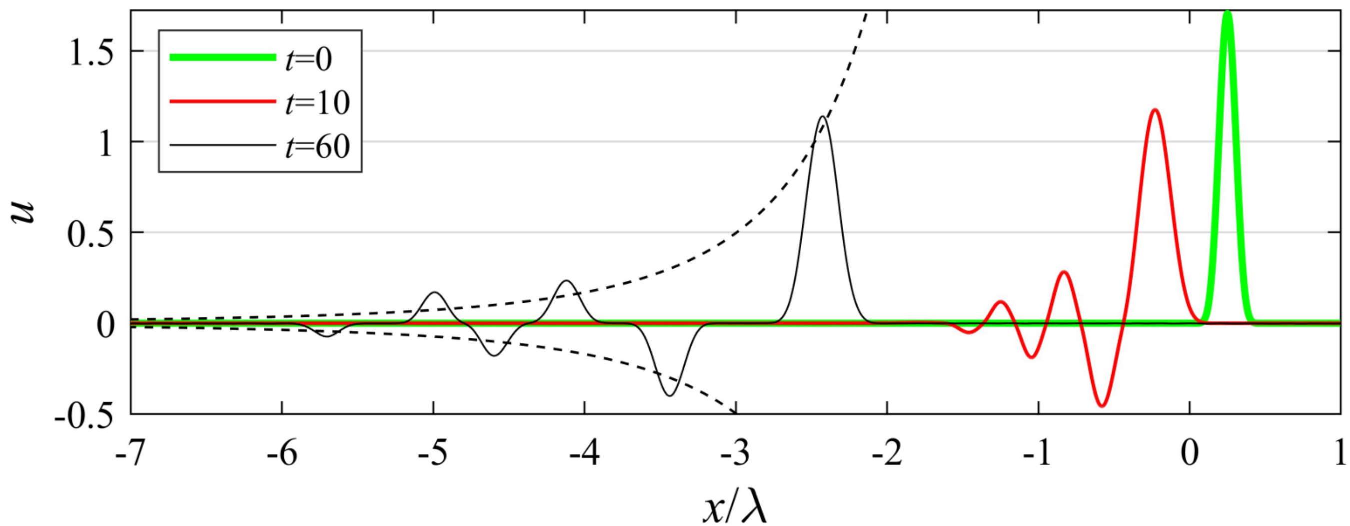
Nothing similar to dispersive tails appears in this simulation. The small-scale compactons play their role.

New compacton solutions

The Cauchy problem for a narrow pulse-like disturbance

Now the initial condition is specified in the form of a compacton with the same amplitude $A \approx 1.7$, but its shape is twice narrower than of the compacton.

The solution at the initial stage resembles much the desintegration of a pulse-like disturbance in the KdV equation (see the red curve).
Later on, a train of compactons with alternating signs forms.

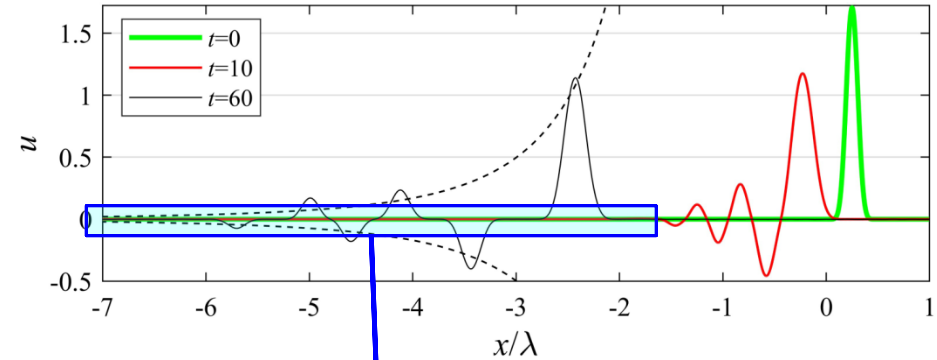


The amplitudes and velocities of the generated solitary waves agree with the relation for compactons (see the dashed curves). The compactons' tops are located to the left from the dashed lines, thus the compactons experience acceleration (the nonlinear shift of coordinates to the left).

New compacton solutions

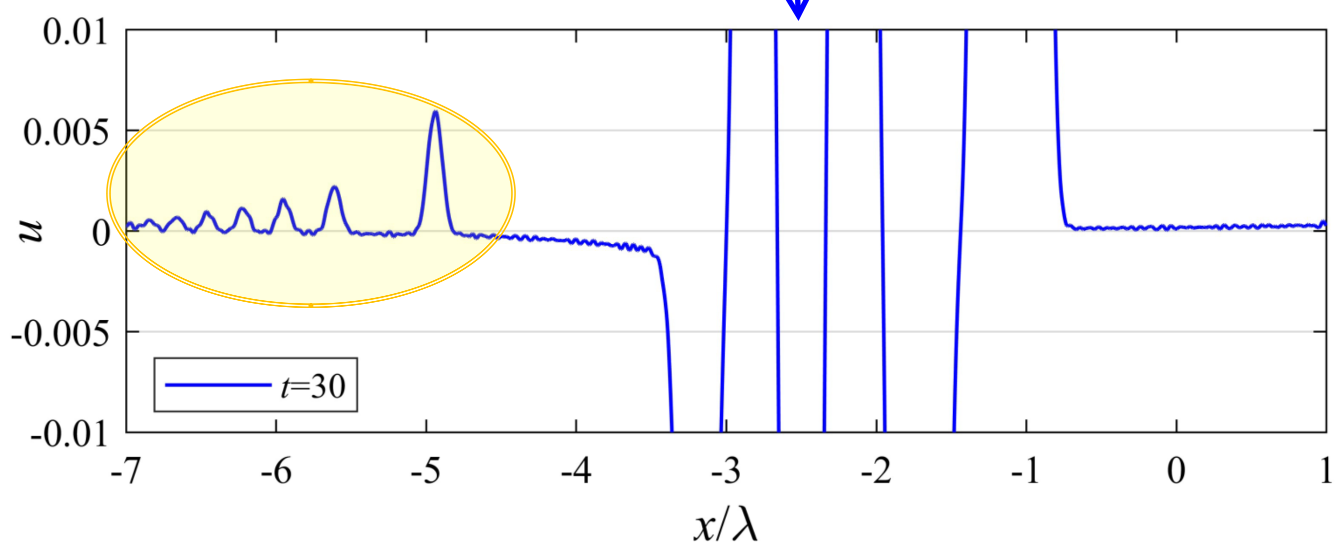
The Cauchy problem for a narrow pulse-like disturbance

Besides the train of alternating compactons of relatively large amplitudes, a sequence of smaller-amplitude compactons appears, which have the signs similar to the initial condition.



The small-amplitude compactons are shorter and faster; they quickly occupy the entire domain of simulations and result in numerous new collisions.

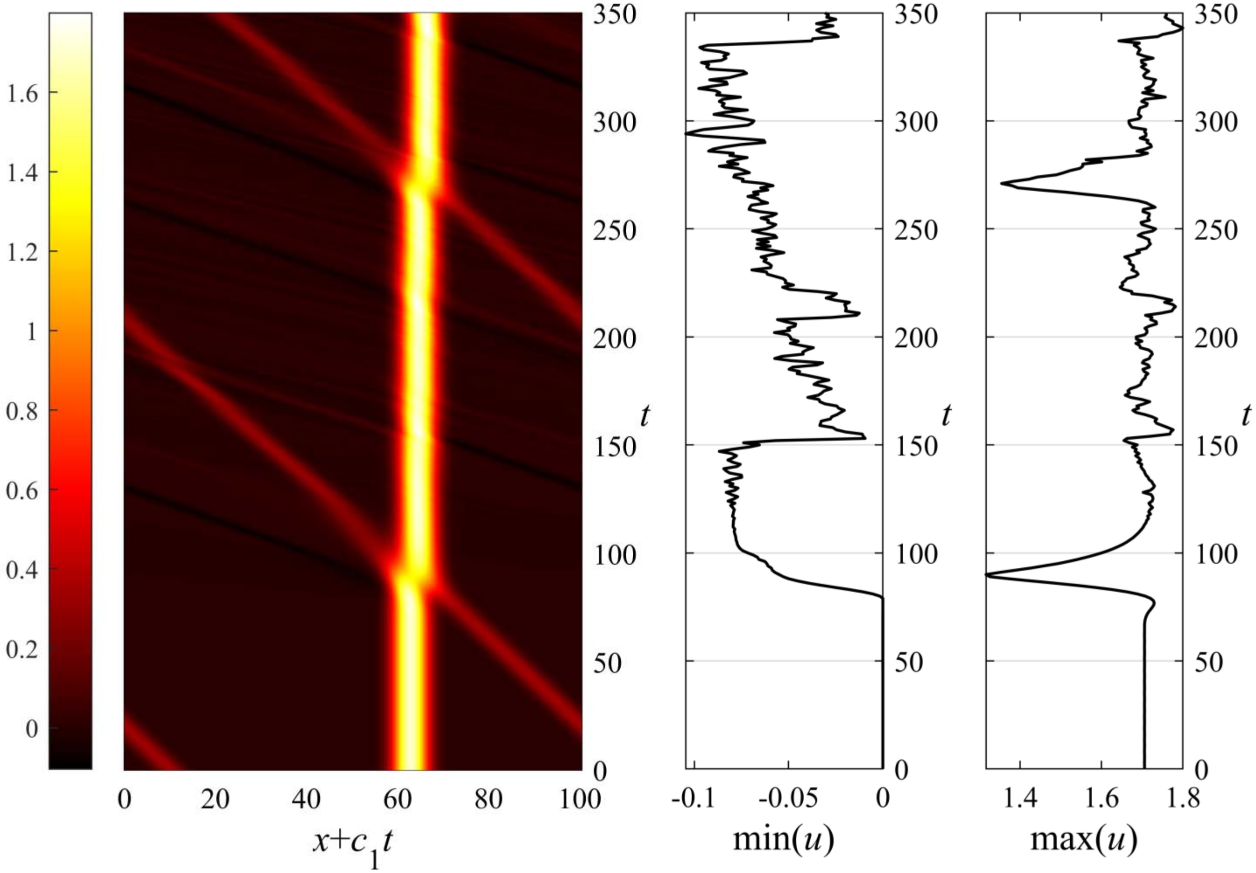
In this simulation the small-amplitude compactons play the role of dispersive waves in the integrable KdV equation.



New compacton solutions

Collisions of compactons with same signs

The initial condition is taken in the form of two distant compactons with the parameters $c = 1$ and $c = 1.5$. Due to the difference in speeds and periodic spatial domain they collide. This process is almost elastic. Some noisy oscillations of both signs appear in the domain, but do not show the tendency to grow in magnitude.



Similar to the integrable systems, during the collision the faster solitary wave accelerates, the slower decelerates.

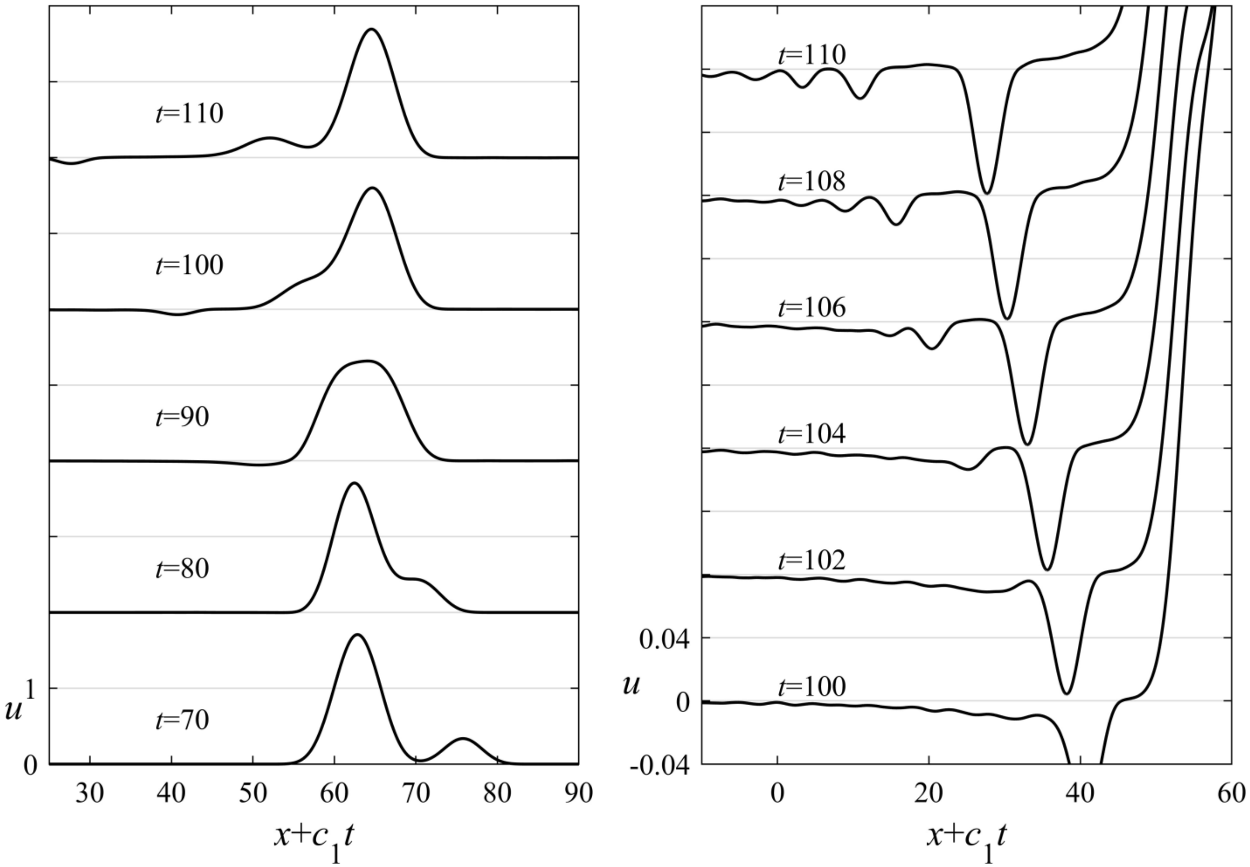
The solution reduces the amplitude, similar to the case of interaction of KdV solitons.

$$A_1 - A_2 \approx 1.4$$

New compacton solutions

Collisions of compactons with same signs

The snapshots of the solution reveal the overtaking type of the interaction.
The interaction of compactons with the same polarity produces small-amplitude compactons of the opposite polarity. Later on the small-amplitude compactons participate in new collisions and produce the noisy wave background.



The solution reduces the amplitude, similar to the case of interaction of KdV solitons.

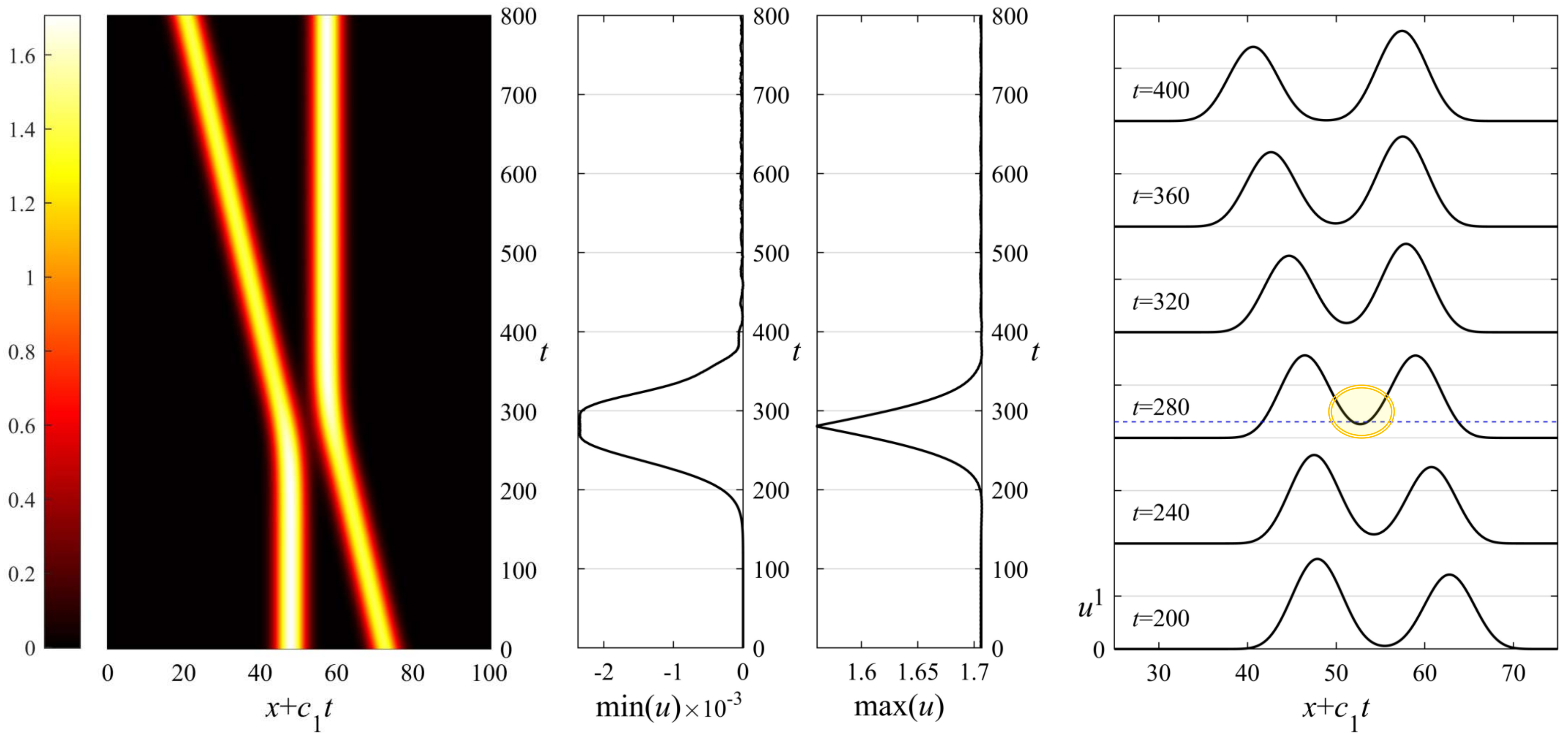
$A_1 - A_2 \approx 1.4$

New compacton solutions

Collisions of compactons with same signs

The exchange-type interaction is observed between the compactons with very close velocities, $c = 1$ and $c = 1.05$. Surprisingly, though this interaction lasts for a longer time, the compactons preserve their amplitudes to a greater extent.

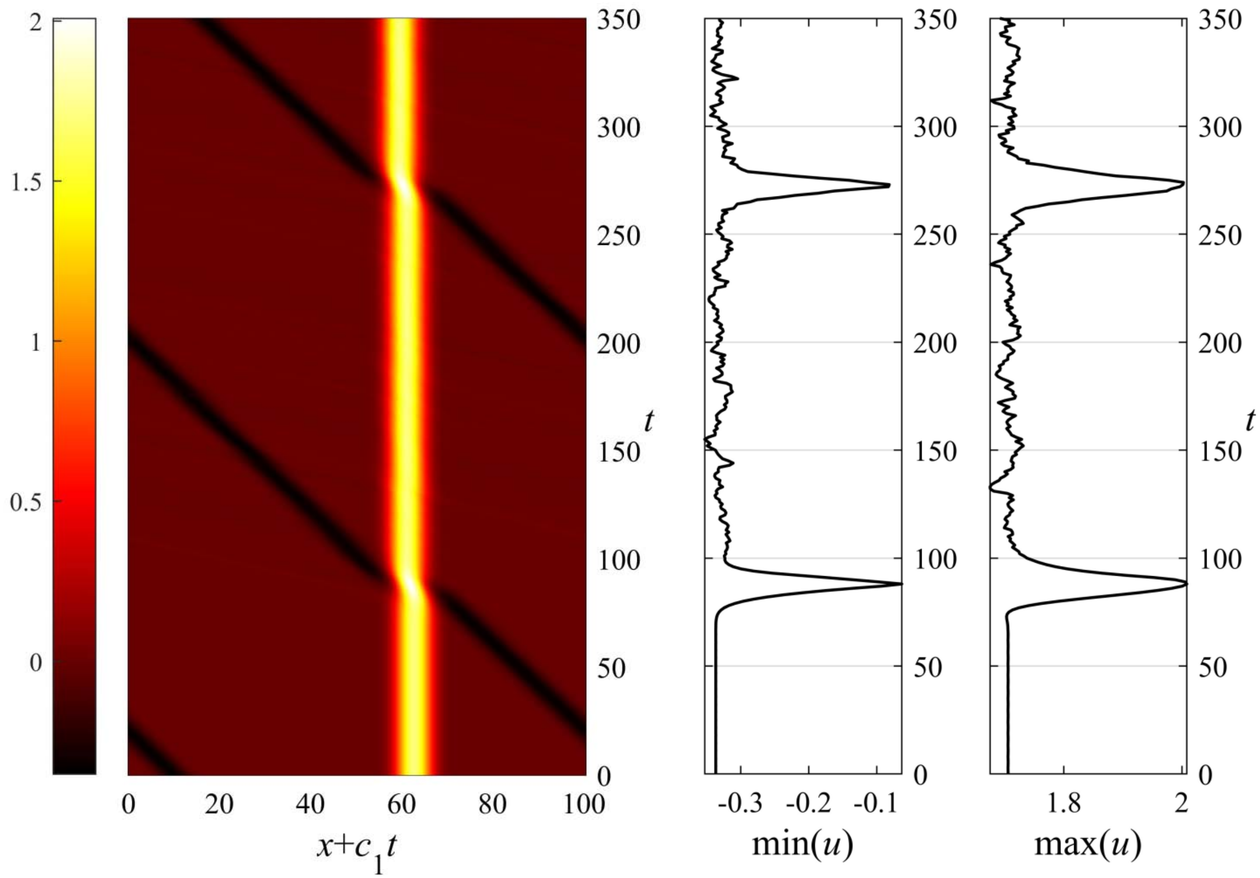
The height of the neck between the compactons is up to about $A_1 – A_2$ (see the horizontal dashed blue line), what agrees with the integrable KdV-type equations.



New compacton solutions

Collisions of compactons with different signs

The initial condition is taken in the form of two distant compactons with the parameters $c = 1$ and $c = 1.5$. Similar to the collisions of solitons with the same signs, the interaction between compactons is close to elastic. The generated noisy waves do not seem to grow in magnitude.



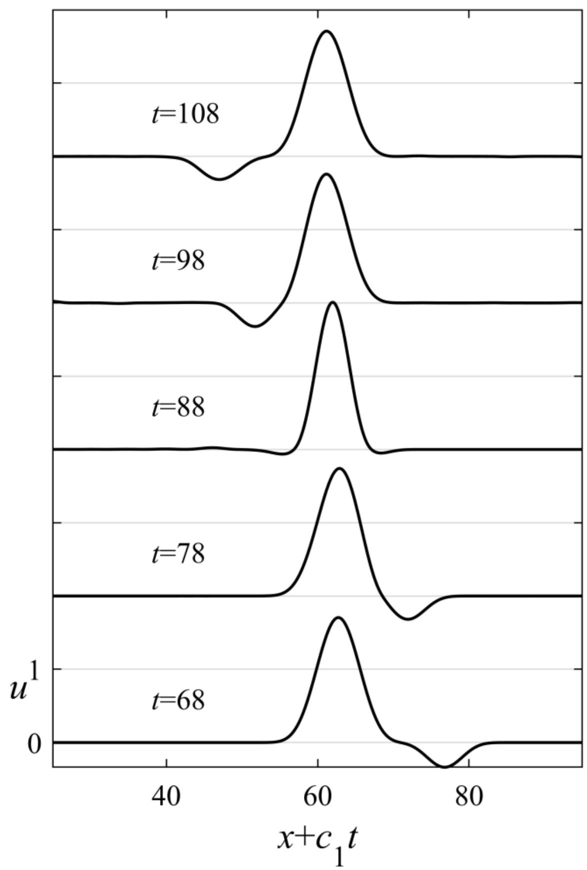
Both compactons speed up during the collision, what is **different** to the conterpart of integrable systems, where the slow soliton decelerates.

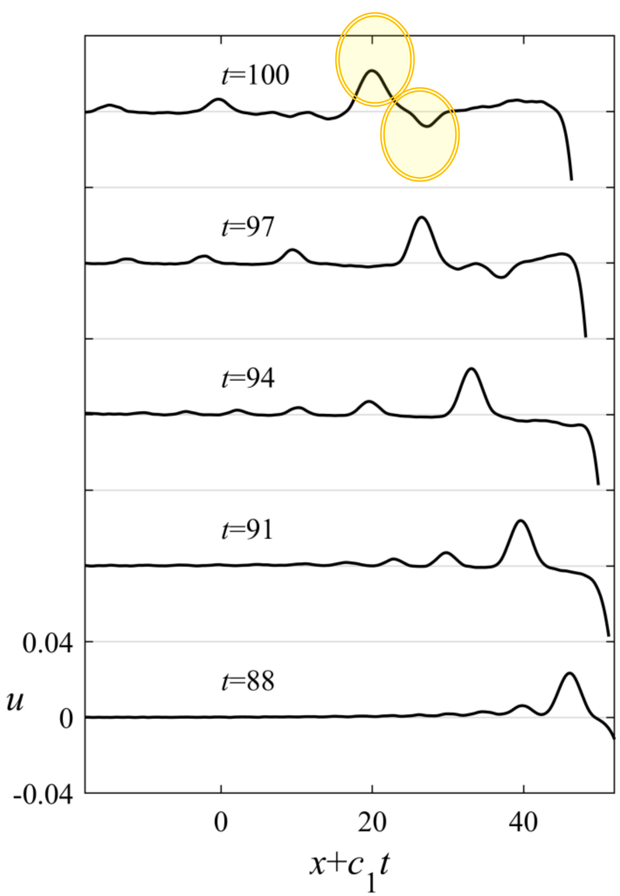
The solution increases the amplitude, similar to the case of absorb-emit interaction of mKdV solitons, $A_1 + A_2 \approx 2.0$

New compacton solutions

Collisions of compactons with different signs

The colliding compactons produce a series of compactons with the sign similar to the larger compacton, though small-amplitude compactons of the other sign appear later on, due to the new collisions.





The solution increases the amplitude, similar to the case of interaction of mKdV solitons, $A_1+A_2 \approx 2.0$.

A the same time, the sign of the cumulative wave within the Gardner equation framework is specified by the faster, not large soliton. The situation is opposite in the present simulation.

Conclusions

1. Compactons of the sublinear KdV equation $\partial_t u + (u|u|^{\alpha-1})_x + u_{xxx}$, $\alpha \in (0,1)$, have been studied. The compactons are larger in amplitude and wider when the speed is smaller. They may be either positive or negative; they propagate to the same direction as the dispersive quasilinear waves of the classic KdV equation. The compactons are energetically stable with respect to compact perturbations with the same support.

2. The compacton dynamics have been studied numerically for the particular case α = 3/4. The long-term solution of the evolution problem for pulse-like disturbances is shown to tend to a sequence of compactons. Compactons with the opposite polarity may arise in the time evolution of the pulse-like initial data.

3. Interacting compactons emit waves in the form of trains of small-scale compactons, but almost recover their amplitudes after collisions. The three types of collisions known for solitons have been observed. Similar to solitons of the KdV-type integrable equation, the maximum wave field decreases when compactons of the same polarity interact, or increases in the case of opposite polarities. Interacting compactons may experience non-classical nonlinear shifts of coordinates.

4. Compactons play a twofold role. On the one hand, compactons behave similar to solitons. On the other hand, they play the role of dispersive waves of the classic KdV equation.

Submitted to *Comm. Nonlin. Sci. Num. Sim.* ArXiv: 2010.02816.