Geometric approach to study of the Navier-Stokes equations

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Abstract

Some examples of exact solutions of the system of equations of flows of the incompressible fluid are obtained . Their properties are discussed.

1 Introduction

The Navier-Stokes system of equations describing the motion of three-dimensional flows of an incompressible fluid has the form

$$\frac{\partial}{\partial t}\vec{U}(\vec{r},t)+(\vec{U}(\vec{r},t)\cdot\vec{\nabla})U(\vec{r},t)-\mu\,\Delta\vec{U}(\vec{r},t)+\vec{\nabla}P(\vec{r},t)=0,$$

$$(\vec{\nabla}\cdot\vec{U}(\vec{r},t))=0, \tag{1}$$

where $\vec{U}(\vec{r},t)=(U(\vec{r},t),\ V(\vec{r},t),\ W(\vec{r},t))$ are the components of velocity, $P(\vec{r},t)$ is the pressure, and μ is the fluid viscosity. Given system of equations has numerous applications in various branch of modern mathematics, physics and technics, but until recently there are a lot of unsolved problems connected with it. Here we propose a new approach to constructing solutions to the system (1), based on the representation functions and their derivatives in parametric form, and then it will be shown how it can be used to obtain new solutions to the system under consideration in both cases $\mu=0$ and $\mu\neq 0$.

In this communication we study the properties of the system (1) from geometric point of view.

Theorem 1 *The* 14*-dim space* D^{14} *in local coordinates* $x_i=[x,y,z,t,\eta,\rho,m,u,v,w,p,\xi,\chi,n]$ *endowed with the Riemann metrics of the form*

$$ds^2=2\,dxdu+2\,dydv+2\,dzdw+(-Uu-Vv-Ww)\,dt^2+2\,dtdp+$$

$$+\Big(u\mu\,U_x-vUV+v\mu\,U_y-wUW+w\mu\,U_z-u\,(U)^2-uP-Up\Big)\,d\eta^2+2\,d\eta d\xi+$$

$$+\Big(-uUV+u\mu\,V_x+v\mu\,V_y-wVW+w\mu\,V_z-v\,(V)^2-vP-Vp\Big)\,d\rho^2+2\,d\rho d\chi+$$

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$$+\left(-uUW+u\mu W_x-vVW+v\mu W_y+w\mu W_z-w\,(W)^2-wP-Wp\right)dm^2+2\,dmdn, \qquad (2)$$

with the components depending from the functions U,V,W,P has components of the Ricci-tensor $R_{44}=0$ on solutions of the system (1).

The metric belongs to the class of the Riemann spaces with vanishing scalar Invariants and part of its geodesics with respect to the coordinates η,ρ,m,ξ,χ,n are determined by the equations

$$\ddot{\eta}=0,\ \ddot{\rho}=0,\ \ddot{m}=0,\ \ddot{\xi}=0,\ \ddot{\chi}=0,\ \ddot{n}=0.$$

and have the form

$$\eta(s)=a_1\,s+b_1,\quad \rho(s)=a_2\,s+b_2,\quad m(s)=a_3\,s+b_3,\quad \xi(s)=a_4\,s+b_4,$$

$$\chi(s)=a_5\,s+b_5,\quad n(s)=a_6\,s+b_6.$$

The equations to the coordinates $[x,y,z,t]$ take the form

$$\frac{d^2}{ds^2}x(s)-1/2\,U\left(\frac{d}{ds}t(s)\right)^2+1/2\,a_1{}^2\mu\frac{\partial}{\partial x}U-1/2\,a_1{}^2\,(U))^2-1/2\,a1^2P-1/2\,a_2{}^2UV+1/2\,a2^2\mu\frac{\partial}{\partial x}V-$$

$$-1/2\,a_3{}^2UW+1/2\,a_3{}^2\mu\frac{\partial}{\partial x}W=0,$$

$$\frac{d^2}{ds^2}y(s)-1/2\,V\left(\frac{d}{ds}t(s)\right)^2-1/2\,a_1{}^2UV+1/2\,a_1{}^2\mu\frac{\partial}{\partial y}U+1/2\,a_2{}^2\mu\frac{\partial}{\partial y}V-1/2\,a_2{}^2\,(V)^2-1/2\,a_2{}^2P-$$

$$-1/2\,a_3{}^2VW+1/2\,a_3{}^2\mu\frac{\partial}{\partial y}W=0,$$

$$\frac{d^2}{ds^2}z(s)-1/2\,W\left(\frac{d}{ds}t(s)\right)^2-1/2\,a_1{}^2UW+1/2\,a_1{}^2\mu\frac{\partial}{\partial z}U-1/2\,a_2{}^2VW+1/2\,a_2{}^2\mu\frac{\partial}{\partial z}V+$$

$$+1/2\,a_3{}^2\mu\frac{\partial}{\partial z}W-1/2\,a_3{}^2\,(W)^2-1/2\,a_3{}^2P=0,$$

$$\frac{d^2}{ds^2}t(s)-1/2\,U\,a_1{}^2-1/2\,V\,a_2{}^2-1/2\,W\,a_3{}^2=0,$$

and the coordinates $[u,v,w,p]$ satisfy the linear system of ODE with respect to the $u^i=[u,v,w,p]$ with the coefficients depended on the (x,y,z,t) and looks as

$$\frac{d^2}{ds^2}u(s)=A_1\,u(s)+B_1\,v(s)+C_1\,w(s)+E_1\,p(s),$$

where

$$A_1=1/2\,a_3{}^2\left(\frac{\partial}{\partial x}U\right)W-1/2\,a_1{}^2\mu\frac{\partial^2}{\partial x^2}U+a_1{}^2U\frac{\partial}{\partial x}U-1/2\,a_2{}^2\mu\frac{\partial^2}{\partial x^2}V+1/2\,a_3{}^2U\frac{\partial}{\partial x}W-$$

$$-1/2\,a_3{}^2\mu\frac{\partial^2}{\partial x^2}W+1/2\,a_2{}^2\left(\frac{\partial}{\partial x}U\right)V+1/2\left(\frac{d}{ds}t(s)\right)^2\frac{\partial}{\partial x}U+1/2\,a_2{}^2U\frac{\partial}{\partial x}V+1/2\,a_1{}^2\frac{\partial}{\partial x}P,$$

$$B_1=a_2{}^2V\frac{\partial}{\partial x}V-1/2\,a_1{}^2\mu\frac{\partial^2}{\partial x\partial y}U-1/2\,a_2{}^2\mu\frac{\partial^2}{\partial x\partial y}V+1/2\,a_2{}^2\frac{\partial}{\partial x}P+1/2\,a_3{}^2\left(\frac{\partial}{\partial x}V\right)W+$$

$$+1/2\,{a_{3}}^{2}V\frac{\partial}{\partial x}W-1/2\,{a_{3}}^{2}\mu\,\frac{\partial^{2}}{\partial x\partial y}W+1/2\,\left(\frac{d}{ds}t(s)\right)^{2}\frac{\partial}{\partial x}V+1/2\,{a_{1}}^{2}\left(\frac{\partial}{\partial x}U\right)V+1/2\,{a_{1}}^{2}U\frac{\partial}{\partial x}V,$$

$$C_{1}={a_{3}}^{2}W\frac{\partial}{\partial x}W+1/2\,{a_{1}}^{2}U\frac{\partial}{\partial x}W-1/2\,{a_{1}}^{2}\mu\,\frac{\partial^{2}}{\partial x\partial z}U+1/2\,{a_{1}}^{2}\left(\frac{\partial}{\partial x}U\right)W-1/2\,a3^{2}\mu\,\frac{\partial^{2}}{\partial x\partial z}W+$$

$$+1/2\,{a_{3}}^{2}\frac{\partial}{\partial x}P+1/2\,\left(\frac{d}{ds}t(s)\right)^{2}\frac{\partial}{\partial x}W+1/2\,{a_{2}}^{2}\left(\frac{\partial}{\partial x}V\right)W+1/2\,a2^{2}V\frac{\partial}{\partial x}W-1/2\,{a_{2}}^{2}\mu\,\frac{\partial^{2}}{\partial x\partial z}V,$$

$$E_{1}=1/2\,{a_{2}}^{2}\frac{\partial}{\partial x}V+1/2\,{a_{1}}^{2}\frac{\partial}{\partial x}U+1/2\,{a_{3}}^{2}\frac{\partial}{\partial x}W,$$

and

$$\frac{d^{2}}{ds^{2}}v(s)=A_{2}\,u(s)+B_{2}\,v(s)+C_{2}\,w(s)+E_{2}\,p(s),$$

$$\frac{d^{2}}{ds^{2}}w(s)=A_{3}\,u(s)+B_{3}\,v(s)+C_{3}\,w(s)+E_{3}\,p(s),$$

$$\frac{d^{2}}{ds^{2}}p(s)=A_{4}\,u(s)+B_{4}\,v(s)+C_{4}\,w(s)+E_{4}\,p(s),$$

where

$$E_{4}=1/2\,{a_{3}}^{2}\frac{\partial}{\partial t}W+1/2\,{a_{2}}^{2}\frac{\partial}{\partial t}V+1/2\,{a_{1}}^{2}\frac{\partial}{\partial t}U,$$

$$C4=1/2\,{a_{3}}^{2}\,(W)^{2}+1/2\,{a_{1}}^{2}\left(\frac{\partial}{\partial t}U\right)W+1/2\,{a_{2}}^{2}VW+1/2\,{a_{3}}^{2}\frac{\partial}{\partial t}P+1/2\,\left(\frac{d}{ds}t(s)\right)^{2}\frac{\partial}{\partial t}W+$$

$$+{a_{3}}^{2}W\frac{\partial}{\partial t}W+1/2\,{a_{1}}^{2}UW+1/2\,{a_{1}}^{2}U\frac{\partial}{\partial t}W-1/2\,{a_{1}}^{2}\mu\,\frac{\partial^{2}}{\partial t\partial z}U-1/2\,{a_{3}}^{2}\mu\,\frac{\partial^{2}}{\partial t\partial z}W-$$

$$-1/2\,{a_{2}}^{2}\mu\,\frac{\partial^{2}}{\partial t\partial z}V+1/2\,{a_{2}}^{2}V\frac{\partial}{\partial t}W+1/2\,{a_{2}}^{2}\left(\frac{\partial}{\partial t}V\right)W,$$

$$B4=1/2\,{a_{1}}^{2}\left(\frac{\partial}{\partial t}U\right)V+1/2\,{a_{2}}^{2}\frac{\partial}{\partial t}P+1/2\,{a_{2}}^{2}\,(V)^{2}-1/2\,{a_{2}}^{2}\mu\,\frac{\partial^{2}}{\partial t\partial y}V+1/2\,{a_{1}}^{2}U\frac{\partial}{\partial t}V+$$

$$+1/2\,\left(\frac{d}{ds}t(s)\right)^{2}\frac{\partial}{\partial t}V+1/2\,{a_{3}}^{2}\left(\frac{\partial}{\partial t}V\right)W+1/2\,{a_{3}}^{2}V\frac{\partial}{\partial t}W+1/2\,{a_{1}}^{2}UV-1/2\,{a_{1}}^{2}\mu\,\frac{\partial^{2}}{\partial t\partial y}U+$$

$$+1/2\,{a_{3}}^{2}VW+{a_{2}}^{2}V\frac{\partial}{\partial t}V-1/2\,{a_{3}}^{2}\mu\,\frac{\partial^{2}}{\partial t\partial y}W,$$

$$A_{4}={a_{1}}^{2}U\frac{\partial}{\partial t}U+1/2\,{a_{1}}^{2}\frac{\partial}{\partial t}P-1/2\,{a_{2}}^{2}\mu\,\frac{\partial^{2}}{\partial t\partial x}V+1/2\,{a_{1}}^{2}\,(U)^{2}+1/2\,{a_{3}}^{2}U\frac{\partial}{\partial t}W-1/2\,{a_{3}}^{2}\mu\,\frac{\partial^{2}}{\partial t\partial x}W+$$

$$+1/2\,{a_{2}}^{2}UV+1/2\,{a_{3}}^{2}\left(\frac{\partial}{\partial t}U\right)W+1/2\,{a_{2}}^{2}\left(\frac{\partial}{\partial t}U\right)V+1/2\,{a_{2}}^{2}U\frac{\partial}{\partial t}V+1/2\,{a_{3}}^{2}UW+$$

$$+1/2\,\left(\frac{d}{ds}t(s)\right)^{2}\frac{\partial}{\partial t}U-1/2\,{a_{1}}^{2}\mu\,\frac{\partial^{2}}{\partial t\partial x}U.$$

2 On the metrics with vanishing scalar Invariants

For the metric (2), all scalar invariants constructed from the Riemann tensor and its covariant derivatives are equal to zero. This circumstance forces us to use invariants of another type - the Cartan invariants

Proposition 1

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K:=R{^i a j b}*R{^j c i d}* m^{a}*m^{b}*m^{c}*m^{d}
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S:=R{a b}*R{c d} n{^a}*n{^b}*n{c}*n{^d}
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As example

$$K=(U(x,y,z,t))^2\frac{\partial}{\partial x}U(x,y,z,t)+\left(\frac{\partial}{\partial z}W(x,y,z,t)\right)(W(x,y,z,t))^2+(U(x,y,z,t))^2\frac{\partial}{\partial z}W(x,y,z,t)+$$

$$+\left(\frac{\partial}{\partial y}V(x,y,z,t)\right)(W(x,y,z,t))^2+(U(x,y,z,t))^2\frac{\partial}{\partial y}V(x,y,z,t)+\left(\frac{\partial}{\partial x}U(x,y,z,t)\right)(W(x,y,z,t))^2-$$

$$-2\left(\frac{\partial}{\partial z}W(x,y,z,t)\right)U(x,y,z,t)W(x,y,z,t)-2\left(\frac{\partial}{\partial x}U(x,y,z,t)\right)U(x,y,z,t)W(x,y,z,t)-$$

$$-2\left(\frac{\partial}{\partial y}V(x,y,z,t)\right)U(x,y,z,t)W(x,y,z,t)$$

Another way to study the properties of the NS system is to use the differential parameters of Beltrami

Proposition 2 *If $f(x^i)$ and $h(x^i)$ are the functions of coordinates of the space then the functions defined as*

$$\Delta_1(f)=g^{ij}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j},\quad \Delta_1(f,h)=g^{ij}\frac{\partial f}{\partial x_i}\frac{\partial h}{\partial x_j},$$

$$\Delta_2(f)=g^{ij}\frac{\partial^2 f}{\partial x_i,x_j}-\Gamma^k_{ij}\frac{\partial f}{\partial x_k}$$

In the case

$$f=\psi(x,y,z,t,\eta,\rho,m,u,v,w,p,\xi,\chi,n)$$

from the Laplace-Beltrami equation

$$\Delta(\psi)=0$$

in particular we find the equation

$$-\left(\frac{\partial^2}{\partial\xi^2}\_F15\right)uP(x,y,z,t)-\left(\frac{\partial^2}{\partial p^2}\_F15\right)W(x,y,z,t)w-\left(\frac{\partial^2}{\partial p^2}\_F15\right)U(x,y,z,t)u-$$

$$-\left(\frac{\partial^2}{\partial\chi^2}\_F15\right)V(x,y,z,t)p-\left(\frac{\partial^2}{\partial\xi^2}\_F15\right)U(x,y,z,t)p-\left(\frac{\partial^2}{\partial\chi^2}\_F15\right)v\,(V(x,y,z,t))^2-$$

$$-\left(\frac{\partial^2}{\partial p^2}\_F15\right)V(x,y,z,t)v-\left(\frac{\partial^2}{\partial\xi^2}\_F15\right)u\,(U(x,y,z,t))^2-\left(\frac{\partial^2}{\partial n^2}\_F15\right)w\,(W(x,y,z,t))^2-$$

$$-\left(\frac{\partial^2}{\partial n^2}\_F15\right)wP(x,y,z,t)-\left(\frac{\partial^2}{\partial n^2}\_F15\right)W(x,y,z,t)p-\left(\frac{\partial^2}{\partial \chi^2}\_F15\right)vP(x,y,z,t)+$$
$$+\left(\frac{\partial^2}{\partial \chi^2}\_F15\right)w\mu\frac{\partial}{\partial z}V(x,y,z,t)+\left(\frac{\partial^2}{\partial \xi^2}\_F15\right)v\mu\frac{\partial}{\partial y}U(x,y,z,t)-\left(\frac{\partial^2}{\partial \xi^2}\_F15\right)vU(x,y,z,t)V(x,y,z,t)+$$
$$+\left(\frac{\partial^2}{\partial \chi^2}\_F15\right)u\mu\frac{\partial}{\partial x}V(x,y,z,t)+\left(\frac{\partial^2}{\partial n^2}\_F15\right)w\mu\frac{\partial}{\partial z}W(x,y,z,t)+\left(\frac{\partial^2}{\partial \xi^2}\_F15\right)u\mu\frac{\partial}{\partial x}U(x,y,z,t)-$$
$$-\left(\frac{\partial^2}{\partial \chi^2}\_F15\right)uU(x,y,z,t)V(x,y,z,t)-\left(\frac{\partial^2}{\partial \xi^2}\_F15\right)wU(x,y,z,t)W(x,y,z,t)+$$
$$+\left(\frac{\partial^2}{\partial \chi^2}\_F15\right)v\mu\frac{\partial}{\partial y}V(x,y,z,t)-\left(\frac{\partial^2}{\partial \chi^2}\_F15\right)wV(x,y,z,t)W(x,y,z,t)-$$
$$-\left(\frac{\partial^2}{\partial n^2}\_F15\right)uU(x,y,z,t)W(x,y,z,t)+\left(\frac{\partial^2}{\partial n^2}\_F15\right)u\mu\frac{\partial}{\partial x}W(x,y,z,t)-$$
$$-\left(\frac{\partial^2}{\partial n^2}\_F15\right)vV(x,y,z,t)W(x,y,z,t)+\left(\frac{\partial^2}{\partial \xi^2}\_F15\right)w\mu\frac{\partial}{\partial z}U(x,y,z,t)+\left(\frac{\partial^2}{\partial n^2}\_F15\right)v\mu\frac{\partial}{\partial y}W(x,y,z,t)-$$
$$-2\left(\frac{\partial}{\partial n}\_F15\right)\_c_7-2\left(\frac{\partial}{\partial \xi}\_F15\right)\_c_5-2\left(\frac{\partial}{\partial \chi}\_F15\right)\_c_6-2\frac{\partial^2}{\partial v\partial y}\_F15-$$
$$-2\frac{\partial^2}{\partial p\partial t}\_F15-2\frac{\partial^2}{\partial u\partial x}\_F15-2\frac{\partial^2}{\partial w\partial z}\_F15=0,$$
where $\_F15 = \_F15((x,y,z,t,u,v,w,p,\xi,\chi,n))$ and the relation of the form

$$(U(x,y,z,t)-W(x,y,z,t))P(x,y,z,t)/\mu=\left(\frac{\partial}{\partial z}U(x,y,z,t)\right)U(x,y,z,t)-W(x,y,z,t)\frac{\partial}{\partial x}V(x,y,z,t)-$$
$$-W(x,y,z,t)\frac{\partial}{\partial x}U(x,y,z,t)-W(x,y,z,t)\frac{\partial}{\partial x}W(x,y,z,t)+\left(\frac{\partial}{\partial z}V(x,y,z,t)\right)U(x,y,z,t)+$$
$$+\left(\frac{\partial}{\partial z}W(x,y,z,t)\right)U(x,y,z,t), \qquad (3)$$

when

$$\psi(x,y,z,t,\eta,\rho,m,u,v,w,p,\xi,\chi,n)=A1(x,y,z,t,p)e^{-\eta-\xi-m-n-\chi-\rho}.$$

With the help of the condition (3) we find as particular case the system of equations

1.

$$\left(\frac{\partial^2}{\partial z^2}\phi(x,y,z,t)\right)\frac{\partial^2}{\partial x^2}\phi(x,y,z,t)-\left(\frac{\partial^2}{\partial x\partial z}\phi(x,y,z,t)\right)^2+F(x,z,t=0,$$
$$\frac{\partial^3}{\partial x\partial t\partial x}\phi(x,y,z,t)+\frac{\partial^3}{\partial z\partial t\partial z}\phi(x,y,z,t)+\left(\frac{\partial}{\partial z}\phi(x,y,z,t)\right)\left(\frac{\partial^3}{\partial x^3}\phi(x,y,z,t)+\frac{\partial^3}{\partial z\partial x\partial z}\phi(x,y,z,t)\right)-$$
$$-\left(\frac{\partial}{\partial x}\phi(x,y,z,t)\right)\left(\frac{\partial^3}{\partial x^2\partial z}\phi(x,y,z,t)+\frac{\partial^3}{\partial z^3}\phi(x,y,z,t)\right)-$$
$$-\mu\left(\frac{\partial^4}{\partial x^4}\phi(x,y,z,t)+2\frac{\partial^4}{\partial z\partial x^2\partial z}\phi(x,y,z,t)+\frac{\partial^4}{\partial z^4}\phi(x,y,z,t)\right)-$$

$$-\mu\left(\frac{\partial^4}{\partial y\partial x^2\partial y}\phi(x,y,z,t)+\frac{\partial^4}{\partial z\partial y^2\partial z}\phi(x,y,z,t)\right)=0,$$

which has solutions depending from the Heat equation

2.

$$U(x,y,z,t)=\frac{y-x+xt-tK(z,t)\sin(t^{-1})}{t^2},$$

$$V(x,y,z,t)=\frac{y-2\,x+yt-tK(z,t)\left(\sin(t^{-1})+\cos(t^{-1})\right)}{t^2},$$

$$W(x,y,z,t)=-2\,\frac{z}{t},$$

and

$$\frac{\partial}{\partial t}K(z,t)+\mu\,\frac{\partial^2}{\partial z^2}K(z,t)+2\,\frac{z\frac{\partial}{\partial z}K(z,t)}{t}.$$

3 Lagrange variables?

(Dryuma V.S. sedov-110@mi.ras.ru, 2017, http://www.mi.ras.ru/index.php?c=conf, 2017, p.14-16),

In Lagrange description of liquid flows the NS equations are

$$\frac{\partial}{\partial a}P(a,b,c,t)=-\left(\frac{\partial^2}{\partial t^2}X(a,b,c,t)\right)\frac{\partial}{\partial a}X(a,b,c,t)-\left(\frac{\partial^2}{\partial t^2}Y(a,b,c,t)\right)\frac{\partial}{\partial a}Y(a,b,c,t)-$$

$$\left(\frac{\partial^2}{\partial t^2}Z(a,b,c,t)\right)\frac{\partial}{\partial a}Z(a,b,c,t)=0,$$

$$\frac{\partial}{\partial b}P(a,b,c,t)=-\left(\frac{\partial^2}{\partial t^2}X(a,b,c,t)\right)\frac{\partial}{\partial b}X(a,b,c,t)-\left(\frac{\partial^2}{\partial t^2}Y(a,b,c,t)\right)\frac{\partial}{\partial b}Y(a,b,c,t)-$$

$$-\left(\frac{\partial^2}{\partial t^2}Z(a,b,c,t)\right)\frac{\partial}{\partial b}Z(a,b,c,t)=0,$$

$$\frac{\partial}{\partial c}P(a,b,c,t)=-\left(\frac{\partial^2}{\partial t^2}X(a,b,c,t)\right)\frac{\partial}{\partial c}X(a,b,c,t)-\left(\frac{\partial^2}{\partial t^2}Y(a,b,c,t)\right)\frac{\partial}{\partial c}Y(a,b,c,t)-$$

$$-\left(\frac{\partial^2}{\partial t^2}Z(a,b,c,t)\right)\frac{\partial}{\partial c}Z(a,b,c,t)=0, \tag{4}$$

$$\left[\begin{array}{ccc}\frac{\partial}{\partial a}X(a,b,c,t) & \frac{\partial}{\partial a}Y(a,b,c,t) & \frac{\partial}{\partial a}Z(a,b,c,t)\\ \frac{\partial}{\partial b}X(a,b,c,t) & \frac{\partial}{\partial b}Y(a,b,c,t) & \frac{\partial}{\partial b}Z(a,b,c,t)\\ \frac{\partial}{\partial c}X(a,b,c,t) & \frac{\partial}{\partial c}Y(a,b,c,t) & \frac{\partial}{\partial c}Z(a,b,c,t)\end{array}\right]=1,$$

or

$$\left(\left(\frac{\partial}{\partial a}X(a,b,c,t)\right)\frac{\partial}{\partial b}Y(a,b,c,t)-\left(\frac{\partial}{\partial b}X(a,b,c,t)\right)\frac{\partial}{\partial a}Y(a,b,c,t)\right)\frac{\partial}{\partial c}Z(a,b,c,t)+$$

$$+\left(-\left(\frac{\partial}{\partial a}X(a,b,c,t)\right)\frac{\partial}{\partial c}Y(a,b,c,t)+\left(\frac{\partial}{\partial c}X(a,b,c,t)\right)\frac{\partial}{\partial a}Y(a,b,c,t)\right)\frac{\partial}{\partial b}Z(a,b,c,t)+$$

$$+\left(\left(\frac{\partial}{\partial c}Y(a,b,c,t)\right)\frac{\partial}{\partial b}X(a,b,c,t)-\left(\frac{\partial}{\partial c}X(a,b,c,t)\right)\frac{\partial}{\partial b}Y(a,b,c,t)\right)\frac{\partial}{\partial a}Z(a,b,c,t)-1=0. \quad (5)$$

The example solution of (4) and (5) is

$$X(a,b,c,t)=Bt\sin(a)\sin(b),\quad Y(a,b,c,t)=At\cos(a)\cos(b),$$

$$Z(a,b,c,t)=-2\,\frac{c}{BAt^2\left(\cos(2\,a)-\cos(2\,b)\right)},\ P(a,b,c,t)=-12\,\frac{c^2}{B^2A^2t^6\left(-\cos(2\,a)+\cos(2\,b)\right)^2}+K(t).$$

OPEN QUESTION!!!:

How to represent the system (4) and (5) in geometric form by help the metrics (2)?