Direct numerical validation of 6-wave 1D kinetic equation.

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1D quintic NLSE.

We consider quintic Nonlinear Schrödinger Equation (NLSE):

$$\mathrm{i} u_t + \Delta u + \mu |u|^4 u = 0,$$

which is relevant in some cases:

- 1D case of NLSE when four waves interactions are non-resonant and one needs to look for further term in intensity.
- superfluid vortices.

Wave Kinetic Equation.

Rigorous derivation of 6-waves kinetic equation for periodic boundary conditions was performed:

$$\frac{\partial n_k}{\partial t} = \gamma \int n_k n_{k1} n_{k2} n_{k3} n_{k4} n_{k5} \left(\frac{1}{n_k} + \frac{1}{n_{k1}} + \frac{1}{n_{k2}} - \frac{1}{n_{k3}} - \frac{1}{n_{k4}} - \frac{1}{n_{k5}} \right) \times$$
$$\times \delta(k + k_1 + k_2 - k_3 - k_4 - k_5) \delta(\omega_k + \omega_{k1} + \omega_{k2} - \omega_{k3} - \omega_{k4} - \omega_{k5}) \times$$
$$\times \mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}k_3 \mathrm{d}k_4 \mathrm{d}k_5$$

Interaction of 3-into-3 waves, energy and number of particles are conserved, two fluxes: of energy and number of particles. KZ-solutions are not realized (wrong sign of fluxes, Fjørtoft argument).

Waves Kinetic Equation. Problem formulation.

Because we performed rigorous derivation of WKE, the following questions naturally arise:

- What choice of system parameters is relevant?
- When we break WKE assumptions?
- How boundary conditions influence the problem?

Let us address the first two questions. We shall perform simulations in both dynamical and WKE frameworks for the same initial conditions.

Initial conditions.

For WKE we consider several harmonics around wavenumber $k = 0$ to be nonzero at the initial moment of time. For dynamical equations we consider the same initial condition, just with random uniformly distributed in the range $[0; 2\pi)$ phase of harmonics. In order to compare with the WKE we average the results of simulation of quintic NLSE over ensemble of 1000 of realization. The only parameter we change is the period of the system L. Nonlinearity coefficient μ is chosen to allow kinetic time to be larger than nonlinear time in the system. We use the following choice:

$$\mu = L^{1.1}.$$

Let us suppose that we would like to consider constant initial condition with amplitude $|a_{0k}| = \sqrt{n_0}$ from $k_{min,init} = -\Delta k N_{max,init}$ till $k_{max,init} = \Delta k N_{max,init}$, with standard definition for wave-numbers grid step: $\Delta k = 2\pi/L$ and zero other harmonics.

Dynamical equations and WKE.

As usual, for dynamical equations we consider periodic boundary conditions and different size of the system. The scaling when all amplitudes are equal to 1 and the width of initial condition is 1 is used. Meaning $k_{min,init} = -1/2$ and $k_{max,init} = 1/2$. Which corresponds to $L = 4\pi N_{max} \simeq 113.09734$, here $N_{max} = 9$. In new rescaled variables kinetic time is 1.0. Number of harmonics is WKE simulations was 81.

Comparison with dynamical equations. $L = 20$.

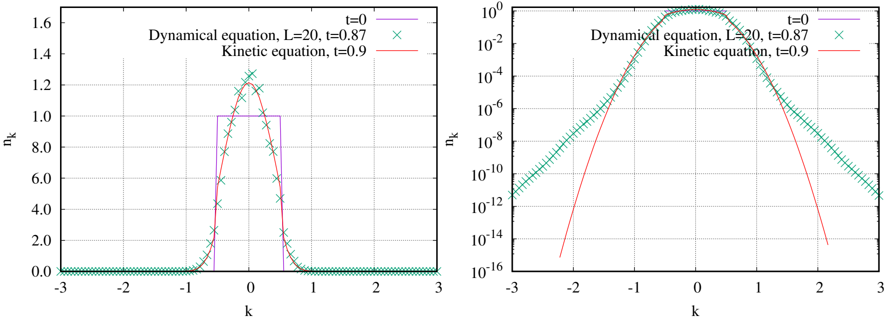


Figure: Comparison of averaged squared amplitudes of harmonics from dynamical simulations with results of simulations in the framework of kinetic equation. Linear and logarithmic scales. Moment of time $t = 0.9$.

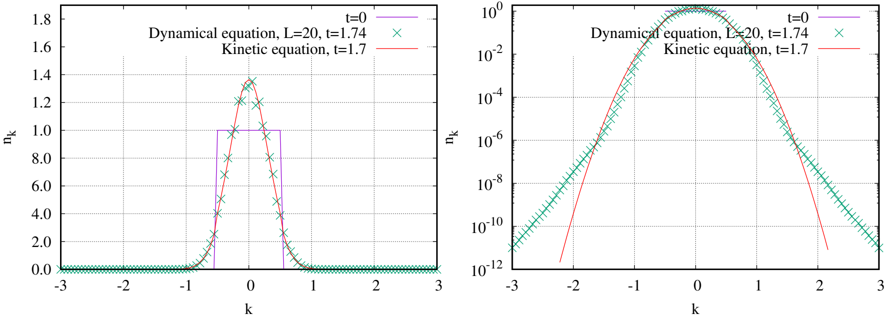


Figure: Comparison of averaged squared amplitudes of harmonics from dynamical simulations with results of simulations in the framework of kinetic equation. Linear and logarithmic scales. Moment of time $t = 1.7$.

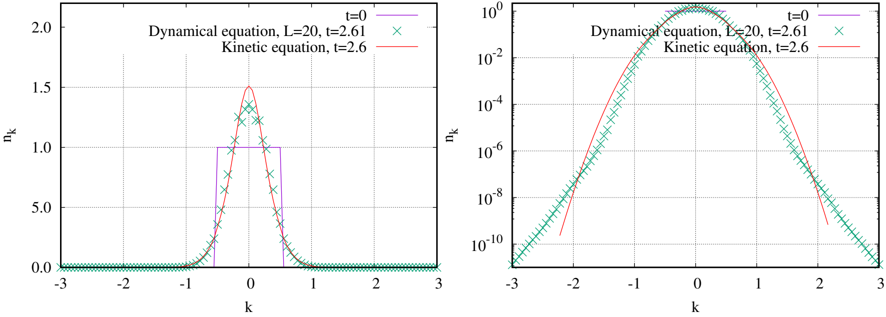
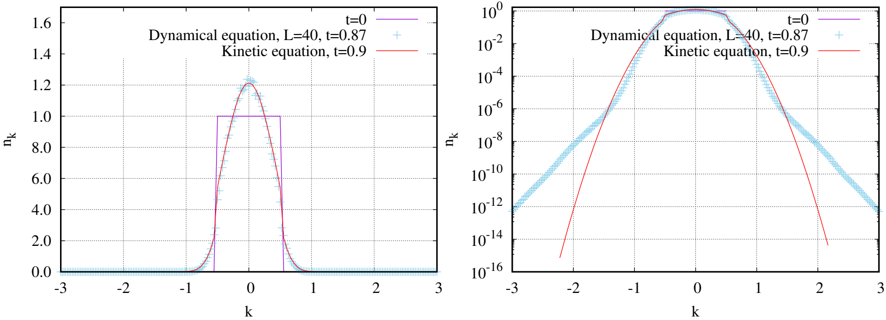


Figure: Comparison of averaged squared amplitudes of harmonics from dynamical simulations with results of simulations in the framework of kinetic equation. Linear and logarithmic scales. Moment of time $t = 2.6$.

Comparison with dynamical equations. $L = 40$, $N_{max} = 19$



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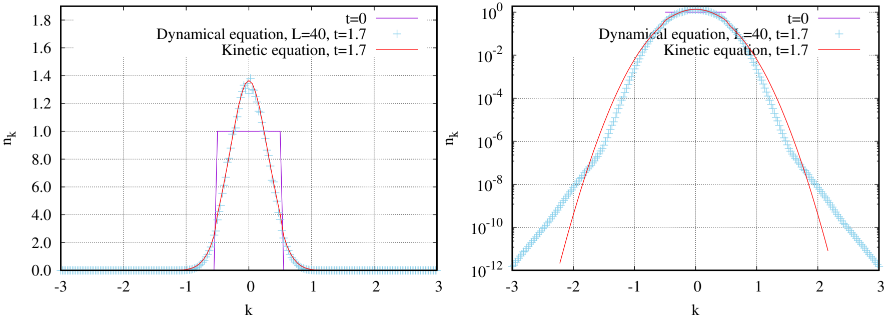

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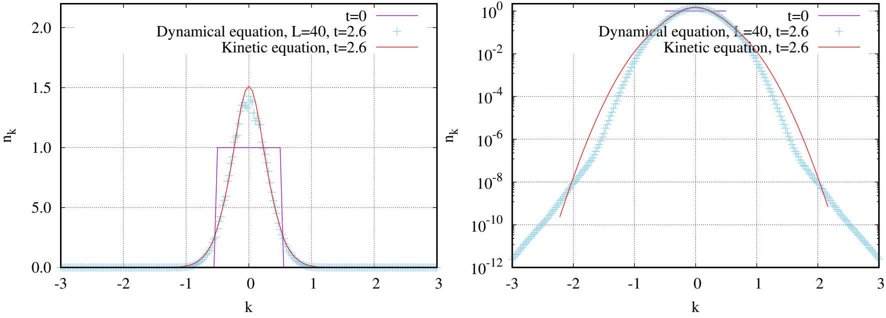

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Comparison of simulations with $L = 40$ and $L = 20$ cases.

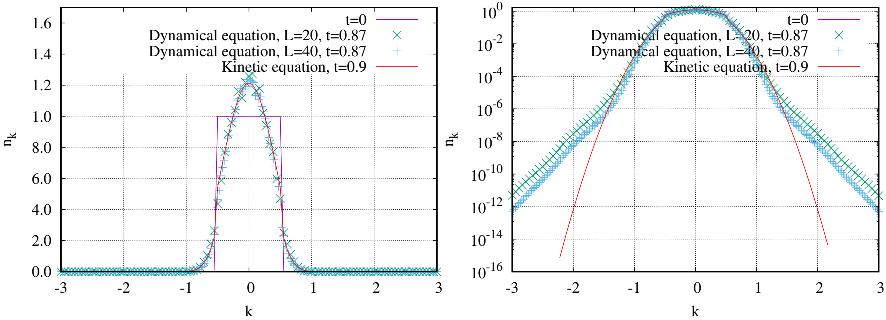
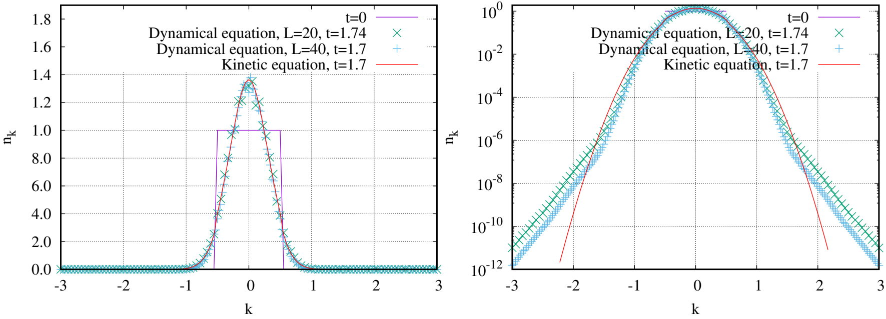


Figure: Comparison of averaged squared amplitudes of harmonics from two dynamical simulations with results of simulations in the framework of kinetic equation. Linear and logarithmic scales. Moment of time $t = 0.9$.



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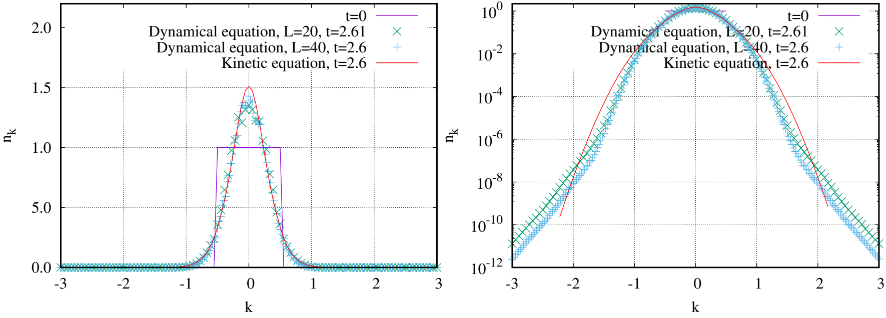
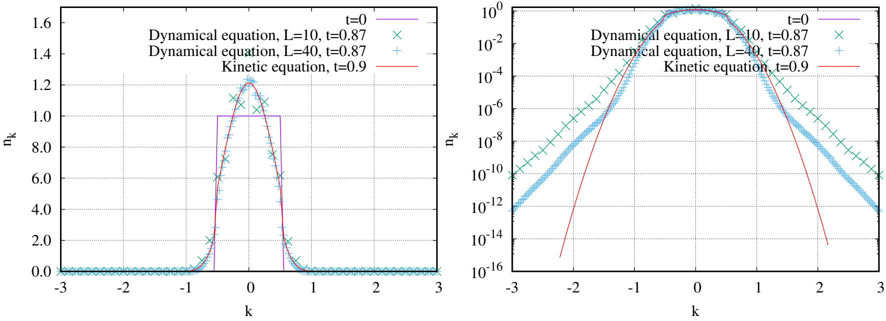


Figure: Comparison of averaged squared amplitudes of harmonics from two dynamical simulations with results of simulations in the framework of kinetic equation. Linear and logarithmic scales. Moment of time $t = 2.6$.

Comparison with dynamical equations. $L = 10$, $N_{max} = 4$



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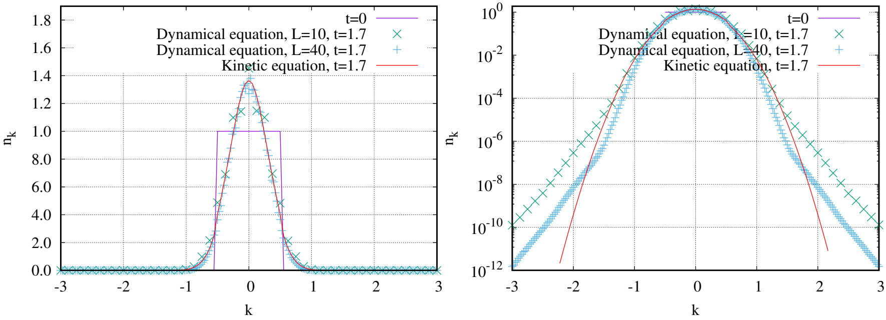

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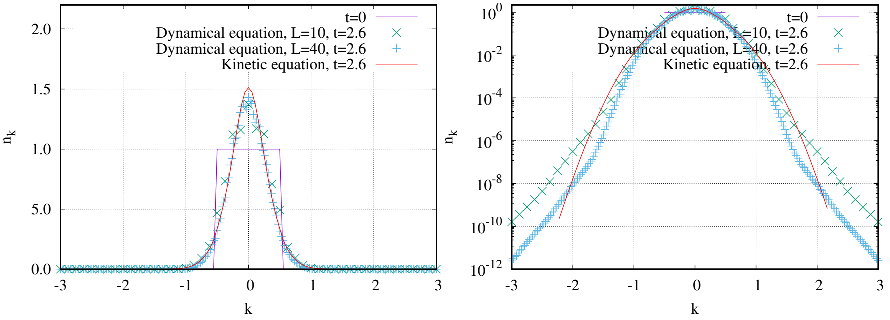

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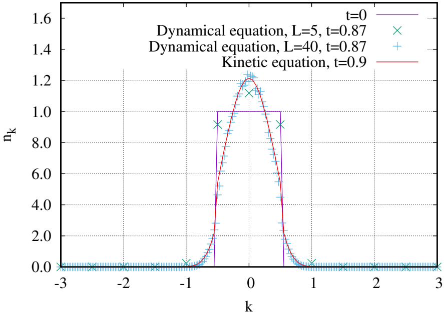
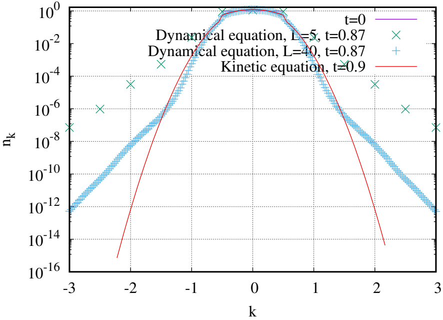
omparison with dynamical equations. $L = 5$, $N_{max} = 1$





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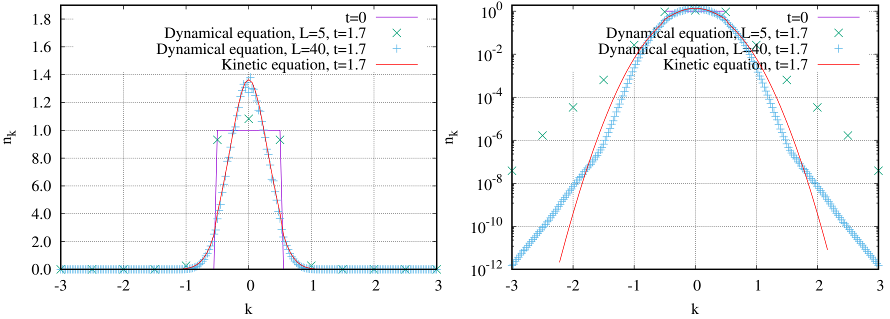

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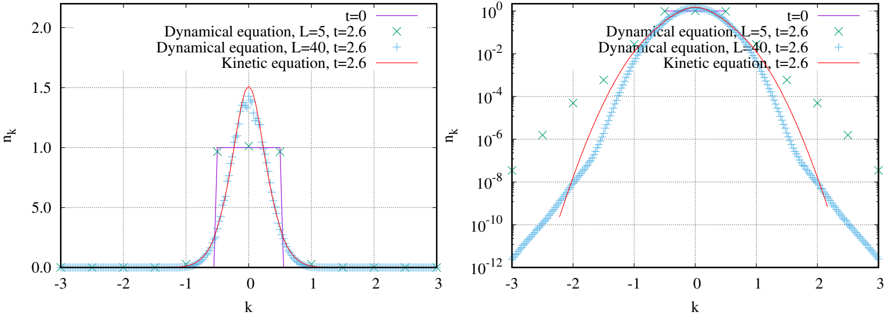

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Results and future work.

- Rigorous derivation of 6-waves WKE for periodic BCs was performed.
- We investigated break up of WKE with the change of the period of the system (step of the wavenumbers grid).
- Future work includes investigation of dependence of the systems behavior with different power of L for μ.
- Direct check of other assumptions of WKE derivation, like pair correlation function etc.
- Influence of different BCs.