Growing of integrable turbulence: new results.

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Introduction.

In this study, we grow turbulence from small initial noise within the 1D-NLSE equation with linear pumping term,

$$i\psi_t + \psi_{xx} + |\psi|^2 \psi = ip_0\psi,$$

until different levels of average intensity $<|\psi|^2>$, and then turn off the pumping and study the resulting integrable turbulence.

How did we come to this problem?

A typical study of turbulence in an integrable system (a.k.a *integrable turbulence*) can be described as follows: (i) we take some integrable system, (ii) we take specific type of initial conditions with some randomness (e.g., modulationally unstable plane wave with random noise), and then (iii) we study evolution of statistics in time, averaging results over random realizations of initial conditions.

Integrable systems allow conservation of infinite series of invariants, so that different types of initial conditions are characterized by different sets of integrals of motion and, during the evolution, demonstrate different statistical behavior even in the long time.

In such studies, it is implicitly assumed that the initial conditions are somehow prepared by an external actor, that resembles a setting of a laboratory experiment. In nature we usually don't have such situation. For instance, when we consider water waves in seas, these waves are "grown" by wind in the same system, where they travel later.

Introduction.

In the present study, we mimic the latter approach: we temporary add a small pumping to an integrable model (making it nonintegrable), wait until the average intensity reaches a certain level, then switch off the pumping and examine the resulting integrable turbulence.

First, we believe that our approach has more resemblance with real physical systems.

Second, one of the phenomena actively studied in integrable turbulence is the statistically stationary state, the existence of which was suggested by V.E. Zakharov in 2009 and later corroborated in numerical simulations for some initial conditions. In these simulations it was necessary to let the system evolve for a very long time, because the initial state is typically far from stationary. In contrast, our approach allows to move consequently through states that are stationary if the pumping is turned off. This setting resembles an ideal gas in a box, in which one molecule is added from time to time.

Formulation of the problem.

We study long-time statistics of solutions to the following dimensionless problem,

$$\psi_{t=0} = \psi_0(x) = A_0 f(x), \quad \overline{|f|^2} = 1,$$

$$\begin{cases} i\psi_t + \psi_{xx} + |\psi|^2 \psi = ip_0\psi, & \text{while} \quad \overline{|\psi|^2} < A_f^2, \\ i\psi_t + \psi_{xx} + |\psi|^2 \psi = 0, & \text{for} \quad \overline{|\psi|^2} = A_f^2, \end{cases}$$

where the function **f(x)** has unit average intensity and unit characteristic spatial scale **δx=1**, the coefficients $\mathbf{A_0}$ and $\mathbf{A_f}$ are the initial and final mean amplitudes, and $\mathbf{p_0}$ is the renormalized pumping coefficient. The overline denotes spatial averaging,

$$\overline{|f|^2} = \frac{1}{L}\int_{-L/2}^{L/2} |f|^2 \, dx,$$

over the simulation box **-L/2 ≤ x < L/2**, with periodic boundary; the period is considered to be large, **L>>1**.

Hence, our results may depend on:

1) initial and final mean amplitudes $\mathbf{A_0}$ and $\mathbf{A_f}$,
2) noise statistics given by the function **f(x)**,
3) pumping coefficient $\mathbf{p_0}$,
4) box size **L**.

Formulation of the problem.

When the pumping is turned off, the equation of motion is the 1D-NLSE of the focusing type, which conserves an infinite series of invariants. The first three of them are wave action (in our notations equals the average intensity), momentum and total energy,

$$N = \overline{|\psi|^2} = \frac{1}{L}\int_{-L/2}^{L/2} |\psi|^2\, dx = \sum_k |\psi_k|^2,$$

$$M = \frac{i}{2L}\int_{-L/2}^{L/2} (\psi_x^* \psi - \psi_x \psi^*)\, dx = \sum_k k\, |\psi_k|^2,$$

$$\mathrm{E} = H_l + H_{nl}, \quad H_l = \frac{1}{L}\int_{-L/2}^{L/2} |\psi_x|^2\, dx = \sum_k k^2\, |\psi_k|^2, \quad H_{nl} = -\frac{1}{2L}\int_{-L/2}^{L/2} |\psi|^4\, dx.$$

Here $\mathbf{H_l}$ is the kinetic energy, $\mathbf{H_{nl}}$ is the potential energy, $\mathbf{k=2\pi m/L}$ is the wavenumber, **m** is integer and $\boldsymbol{\psi_k}$ is the Fourier-transformed wavefield.

When the pumping is on, the integrals evolve with time,

$$dN/dt = 2p_0 N,$$

$$dM/dt = 2p_0 M,$$

$$dE/dt = 2p_0 (E + H_{nl}),$$

...

Formulation of the problem.

Evolution of wave action **N** and momentum **M** is exponential and depends only on their initial values; the time for turning off the pumping is determined as $\mathbf{t_0 = ln[A_f / A_0]/p_0}$.

Evolution of the total energy is less trivial. If the initial noise has small amplitude $\mathbf{A_0 << 1}$, then at the early growth stage the potential energy is small compared to the kinetic one, $\mathbf{H_{nl} << H_l}$. Then, at this stage, the total energy grows exponentially, $\mathbf{E \propto exp(2p_0t)}$. At later stages, when the kinetic and potential energies become comparable, evolution of the total energy depends strongly on their interplay. The behavior of the next-order integrals is expected to be even more complex.

Hence, after turning off the pumping, all realizations of the statistical ensemble will have equal values of wave action and momentum, $\mathbf{N_0exp(2p_0t)}$ and $\mathbf{M_0exp(2p_0t)}$, but should have different values of total energy and the next-order integrals of motion.

In this study we focus on the adiabatic turbulence growth which goes through the consequent stationary states of the integrable turbulence.

We may achieve this when (i) the pumping is small, so that the motion is governed primarily by the terms of the 1D-NLSE, and (ii) the initial state is already close to stationary. The latter is possible if the initial wavefield is almost linear (e.g. small noise), because the linear turbulence is stationary.

Simulation parameters.

Compare the characteristic time scales:

- Pumping: wave action evolves as $\mathbf{N_0 exp(2p_0t)}$, hence $\mathbf{t_p = 1/p_0}$.

- Dispersion: $\mathbf{t_l = \mathit{l}^2}$, where $\mathbf{\mathit{l}}$ is the characteristic length scale of function $\boldsymbol{\psi}$. Our experiments indicate that at the final time it is of the same order as at the initial time, i.e., $\mathbf{\mathit{l}\sim 1}$ and $\mathbf{t_l \sim 1}$ at all times.

- Nonlinearity: $\mathbf{t_{nl} = 1/N}$, where wave action N changes from $\mathbf{A_0^2}$ to $\mathbf{A_f^2}$.

Hence, the adiabatic turbulence growth should be implemented when (i) $\mathbf{t_l} << \mathbf{t_p}$ and $\mathbf{t_{nl} << t_p}$ (evolution is governed primarily by terms of the 1D-NLSE) and (ii) $\mathbf{A_0^2} << 1$ (evolution starts from small noise, which represents almost linear, and therefore almost stationary state). The latter is satisfied for all times if

$$1 \ll \frac{1}{A_0^2} \ll \frac{1}{p_0} \quad \leftrightarrow \quad p_0 \ll A_0^2 \ll 1.$$

If the difference between $\mathbf{p_0}$ and **1** is larger enough in orders of magnitude, then the condition $\mathbf{p_0 << A_0^2}$ is optional. Indeed, if we start from smaller initial noise, $\mathbf{A_0^2} << \mathbf{p_0}$ or $\mathbf{A_0^2} \sim \mathbf{p_0}$, then there is an intermediate time **t*** for which wave action satisfies the full condition, $\mathbf{p_0 << A_0^2 \, exp(2p_0t^*) << 1}$. The equation of motion on the time interval **[0, t*]** is almost linear, so that nonlinear correlation is absent and the state at **t*** is still linear.

Simulation parameters.

At the growth stage, it is also instructive to consider the equation of motion in the Fourier space,

$$\begin{cases} i\dfrac{\partial \psi_k}{\partial t} - k^2\psi_k + (|\psi|^2\psi)_k = ip_0\psi_k, & \text{for} \quad k \neq 0, \\ i\dfrac{\partial \psi_k}{\partial t} + (|\psi|^2\psi)_k = ip_0\psi_k, & \text{for} \quad k = 0. \end{cases}$$

This equation depends explicitly on the wavenumber k. For adiabatic turbulence growth, we need to ensure that the pumping term is much smaller than all other terms present in the equation, including for the smallest nonzero wavenumbers **k**. This leads to condition

$$p_0 \ll \Delta k^2,$$

where **Δk=2*π*/L** is distance between neighbor wavenumbers.

Thus, the conditions for adiabatic turbulence growth are:

$$A_0^2 \ll 1, \quad p_0 \ll 1, \quad p_0 \ll \Delta k^2.$$

The third condition is much more strict than the second one, because we assume **L>>1**. Our experiments indicate that when these conditions are met, the turbulence indeed grows adiabatically.

Numerical methods.

We use Runge-Kutta 4th order method on adaptive grid combined with Fourier interpolation between the grids. For more accurate simulation of the growth stage, we rewrite the 1D-NLSE for function $\boldsymbol{\rho=\exp(-p_0 t)\psi}$, eliminating the right-hand side of the equation. We have checked that, after turning off the pumping, the first ten integrals of motion of the 1D-NLSE are conserved by our numerical scheme up to the relative errors from $\mathbf{10^{-10}}$ (the first three invariants) to $\mathbf{10^{-6}}$ (the tenth invariant) orders.

As initial conditions, we take small noise, $\mathbf{A_0 \ll 1}$, with super-Gaussian Fourier spectrum,

$$f(x) = \sum_k \left(\frac{G_n}{L}\right)^{1/2} e^{-|k|^n + ikx + i\phi_k},$$

where **n** is the exponent defining the shape of the Fourier spectrum, $\boldsymbol{\varphi_k}$ are random phases for each **k** and each realization of the initial conditions, $\mathbf{G_n = \pi\, 2^{1/n} / \Gamma(1+1/n)}$ is the normalization constant such that **f(x)** have unit mean intensity, and **Γ** is Gamma-function. For each of the several experiments presented in the study, we perform simulation for an ensemble of several hundreds of random realizations of phases $\boldsymbol{\varphi_k}$ and then average the results over these realizations. Ensemble size is 200, and we have checked that larger ensemble sizes lead to the same statistical results.

Numerical methods.

After turning off the pumping, we start measurement of the statistical functions, averaging them over the ensemble of random realizations of the initial noise. We examine the ensemble-averaged kinetic **$\langle H_l \rangle$** and potential **$\langle H_{nl} \rangle$** energies, the fourth-order moment of amplitude (a.k.a kurtosis) **k_4**, the PDF **P(I,t)** of relative wave intensity **$I = |\psi|^2/\langle|\psi|^2\rangle$**, the wave-action spectrum,

$$S_k(t) = \langle |\psi_k|^2 \rangle / \Delta k,$$

where **$\Delta k = 2\pi / L$** is the distance between neighbour wavenumbers, and the autocorrelation of the intensity,

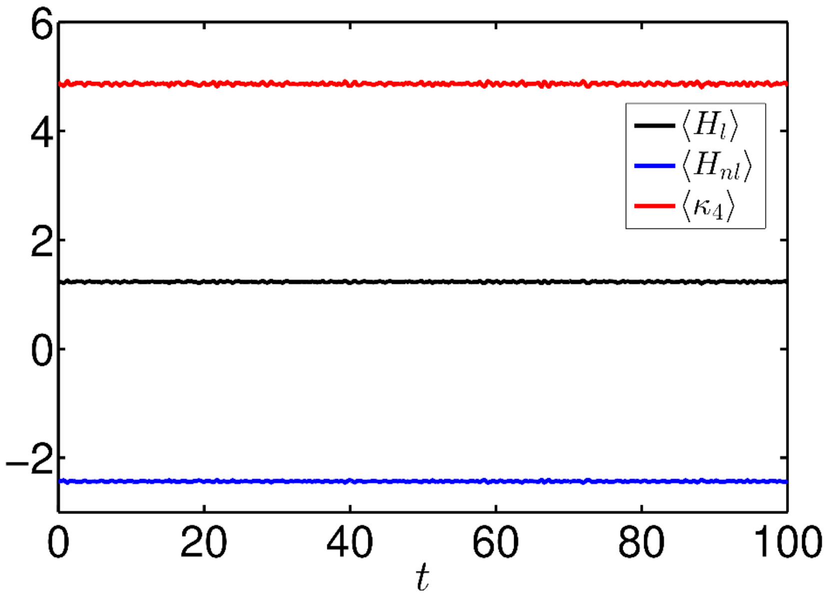
$$g_2(x,t) = \frac{\overline{\langle |\psi(y+x,t)|^2 \cdot |\psi(y,t)|^2 \rangle}}{\langle \overline{|\psi(y,t)|^2} \rangle^2}.$$

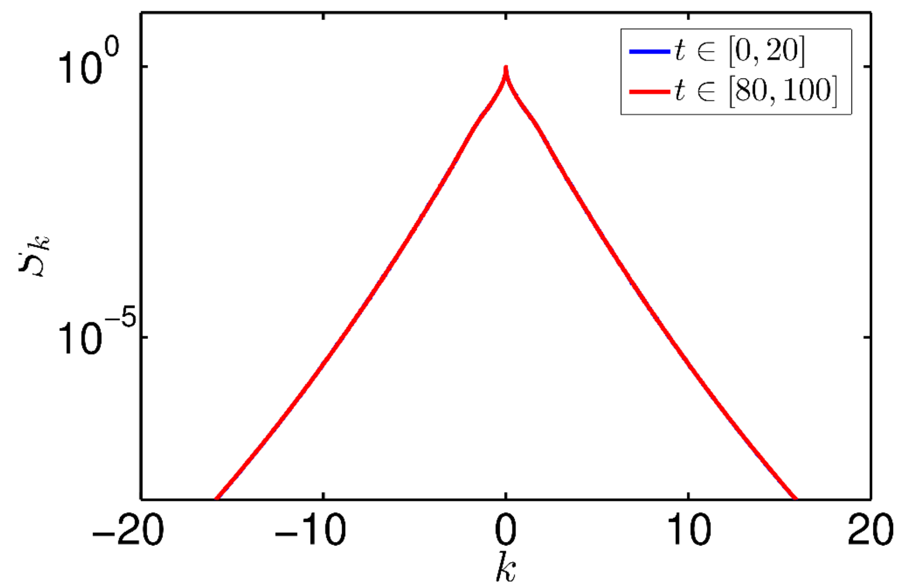
In the latter relation, the overline denotes spatial averaging over the **y** coordinate. Note that, at **x=0**, the autocorrelation equals to the fourth-order moment, **$g_2(0,t)=k_4(t)$**, and at **|x|** → ∞ it must approach to unity, **$g_2(x,t) \to 1$**. For the wave-action spectrum and the PDF, we use normalization conditions

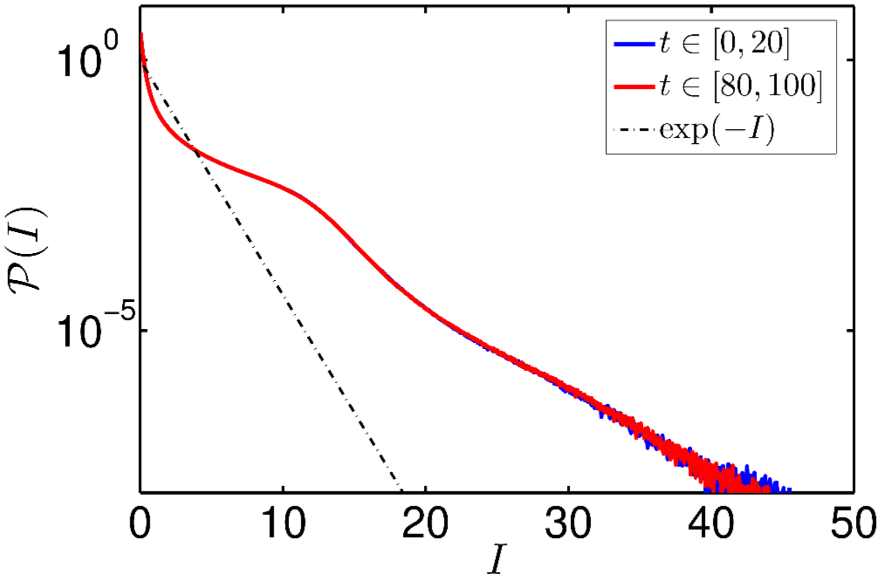
$$\int S_k \, dk = N, \qquad \int \mathcal{P}(I) dI = 1.$$

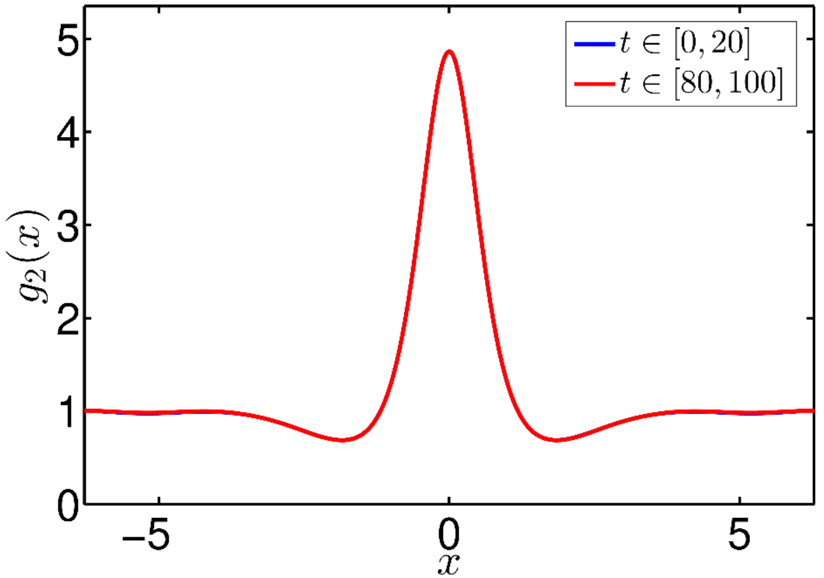
After turning off the pumping, we continue simulations for some time and check that we arrive to the stationary state by comparing statistical functions averaged over time intervals **[0,20]** (“short”) and **[80,100]** (“long”).

Results: stationary state.



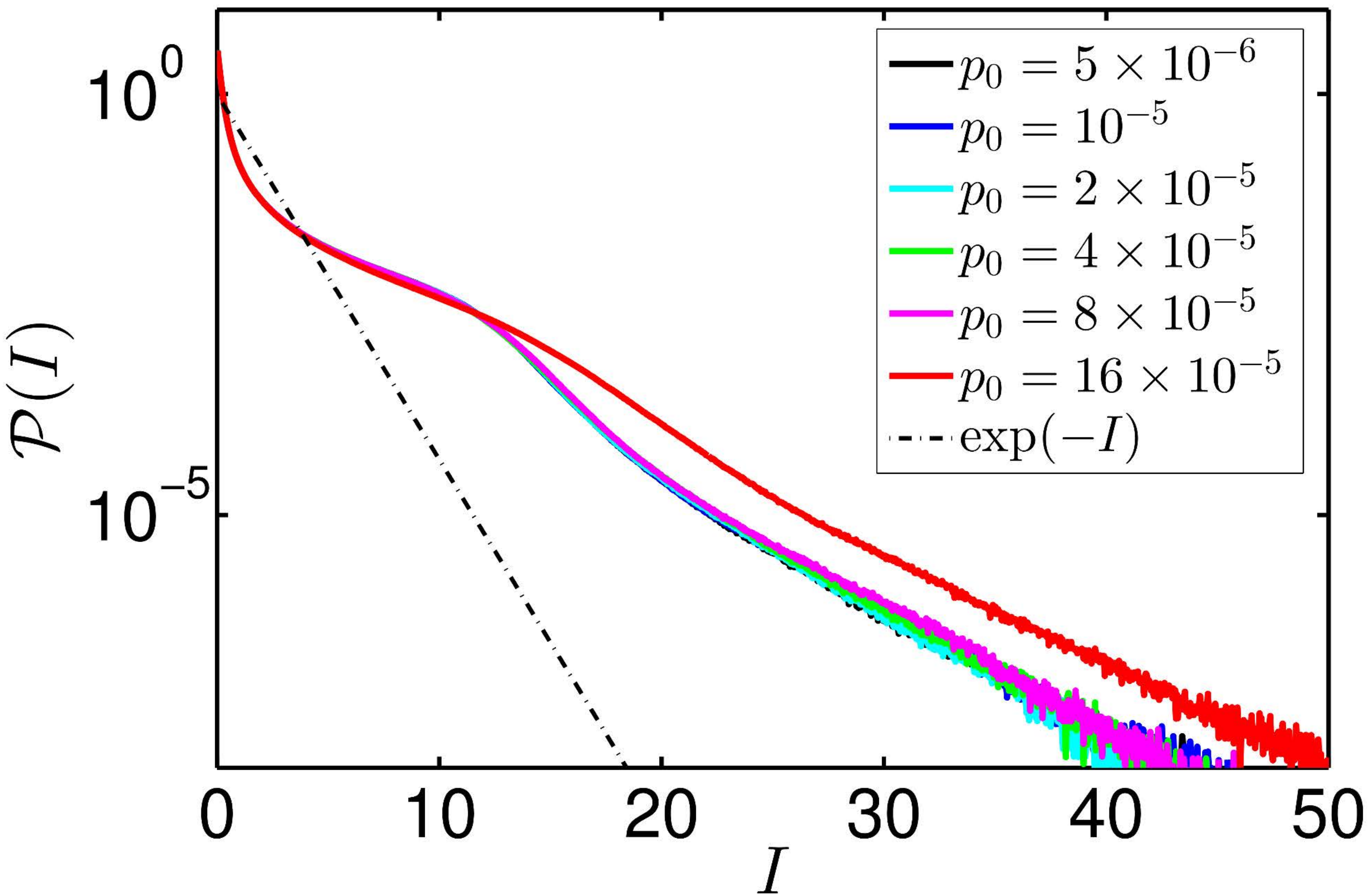






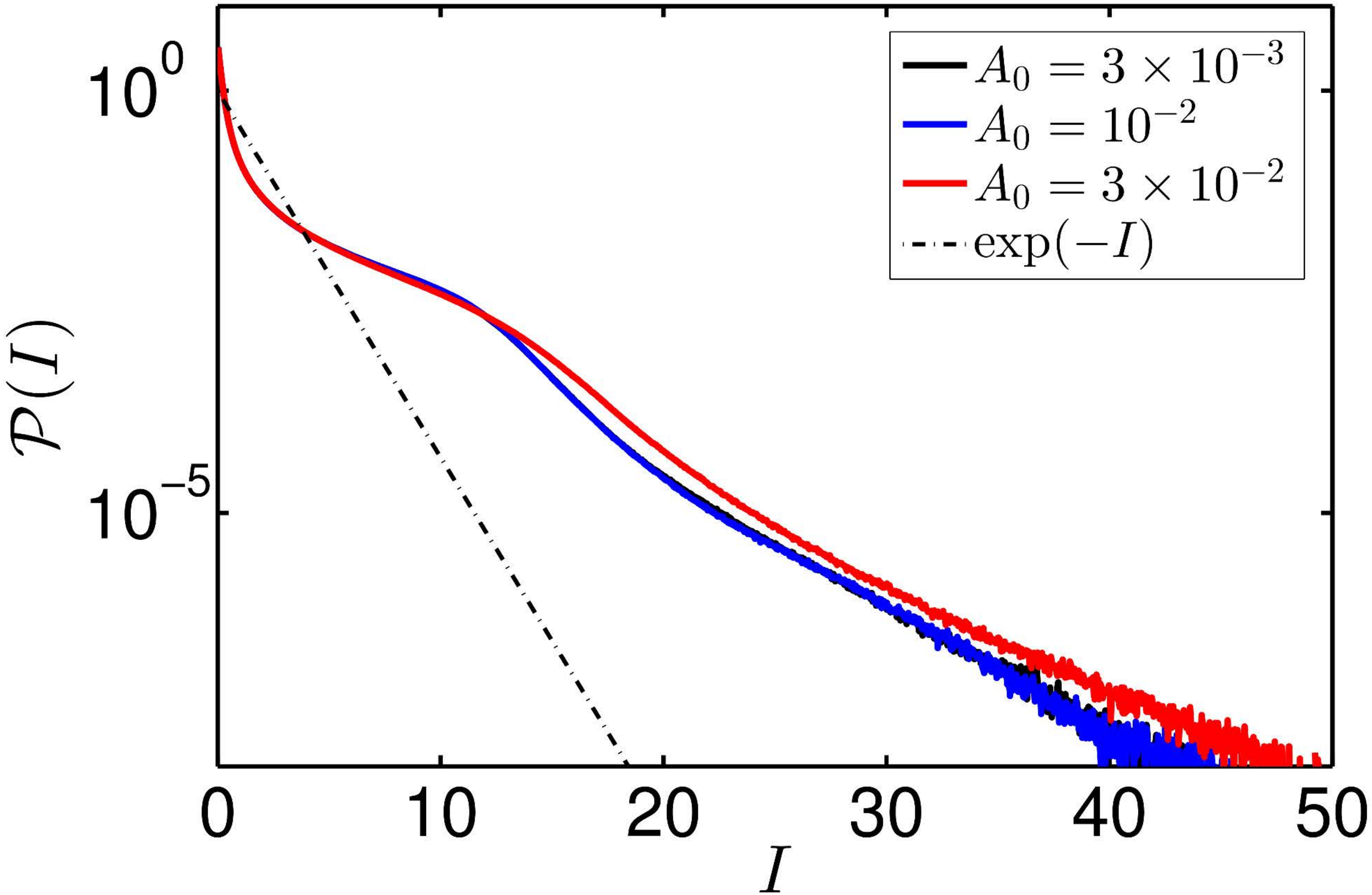
$L = 128\pi$, $n = 2$, $A_0 = 10^{-2}$, $A_f = 1$, $p_0 = 10^{-5}$.

Results: adiabatic regime.



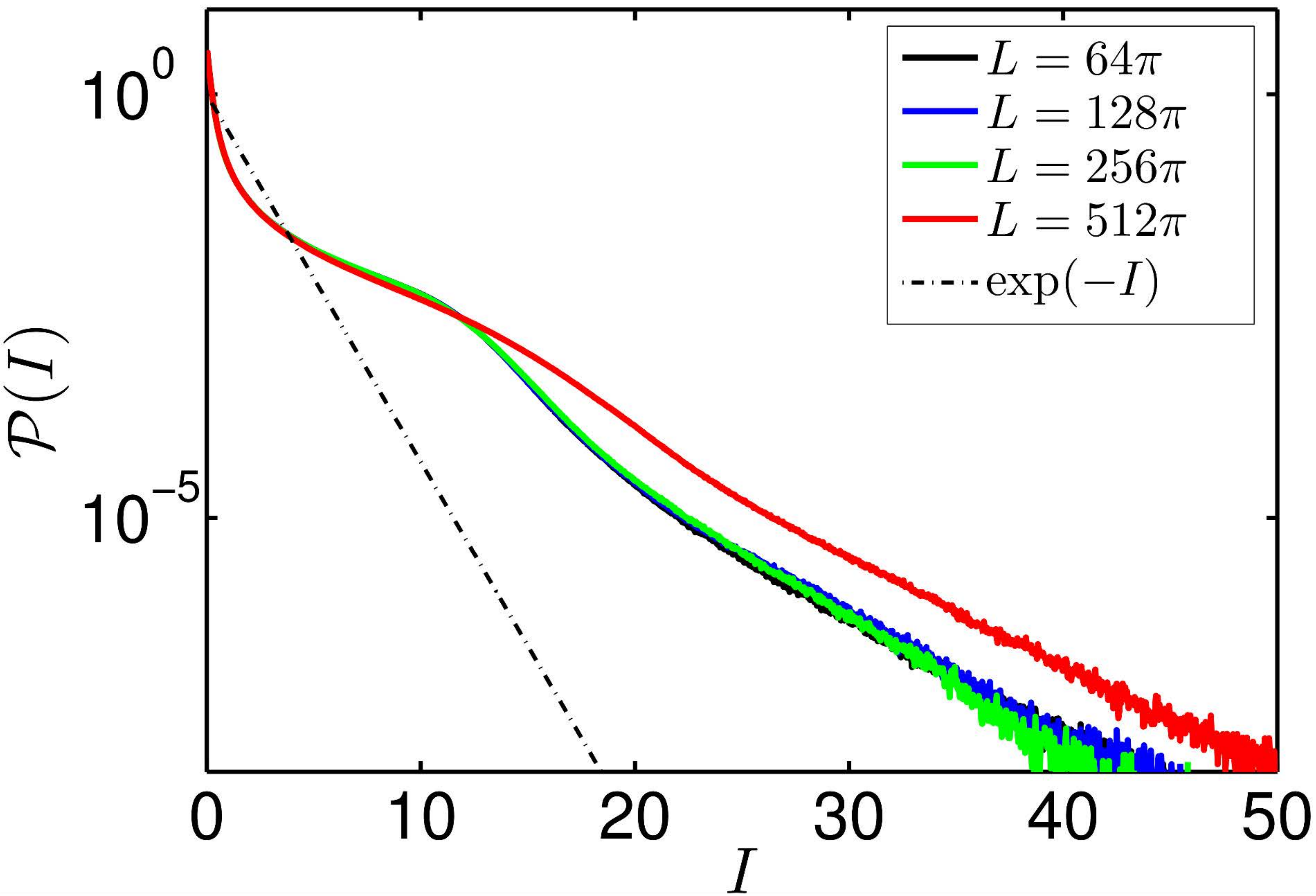
L = 128π, n = 2, $A_0 = 10^{-2}$, $A_f = 1$, comparison of different pumping coefficients p_0. The difference appears exactly when the criterion $\mathbf{p_0 << \Delta k^2}$ ceases to work. Same for other L.

Results: adiabatic regime.



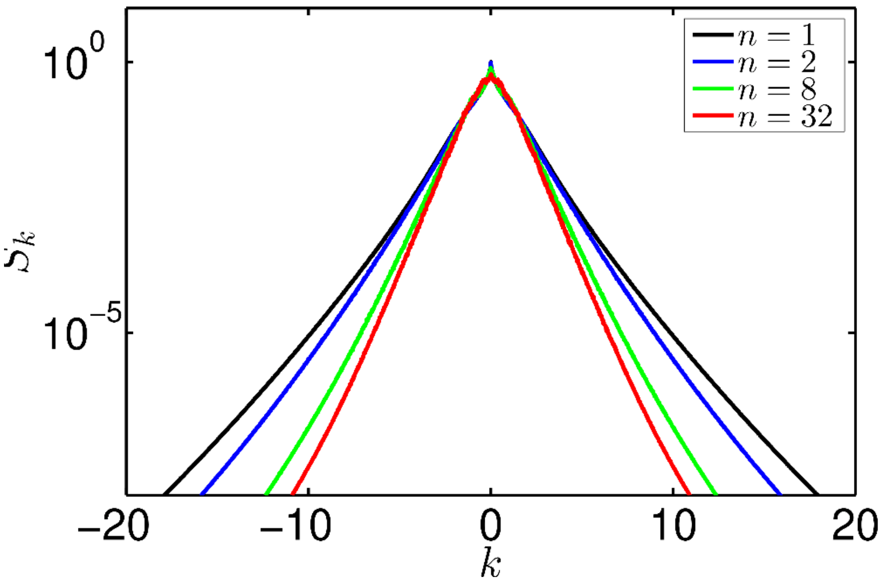
L = 128π, n = 2, A_f = 1, $p_0 = 10^{-5}$, comparison of different initial amplitudes A_0.
No dependency for $A_0 \leq 10^{-2}$.

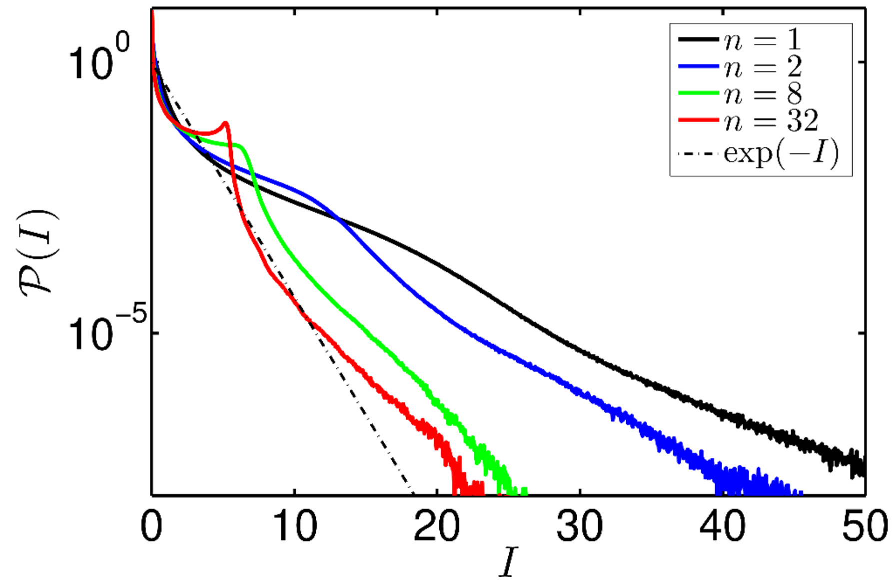
Results: adiabatic regime.

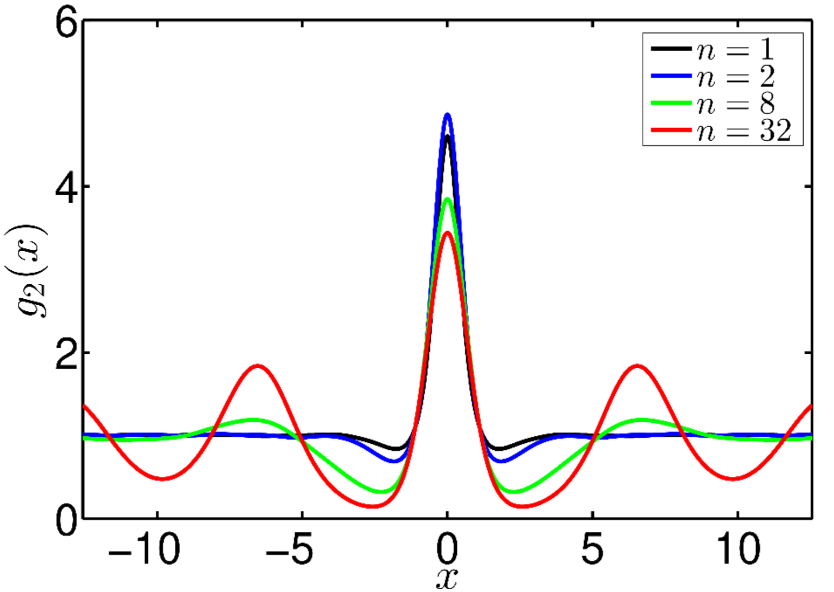


n = 2, $A_0 = 10^{-2}$, $A_f = 1$, $p_0 = 10^{-5}$, comparison of different basin lengths L.
No dependency for $L \leq 256\pi$, i.e., until the criterion $\mathbf{p_0 \ll \Delta k^2}$ works.

Results: dependency on initial spectrum.





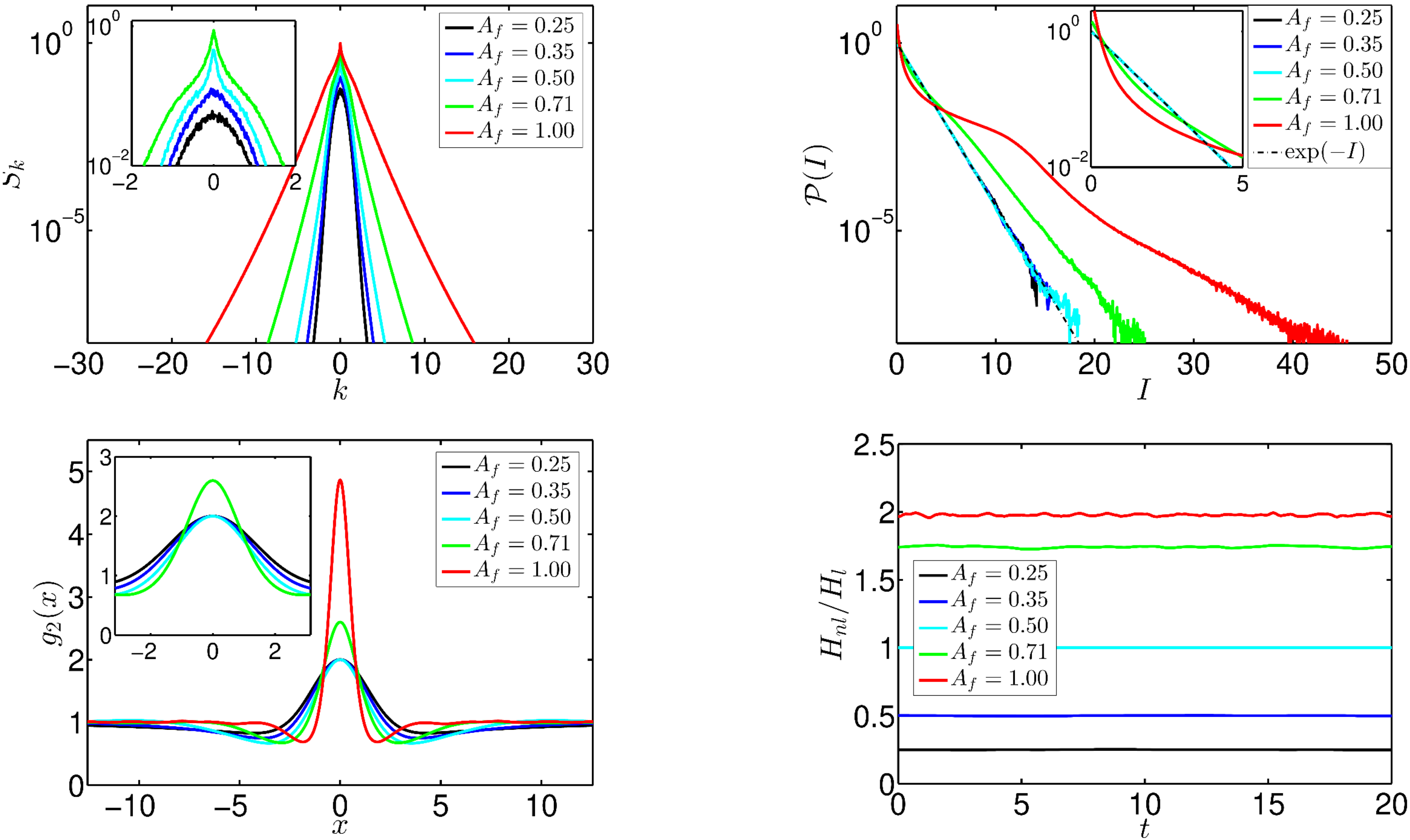


L = 128π, $A_0 = 10^{-2}$, $A_f = 1$, $p_0 = 10^{-5}$, comparison of different exponents **n** defining the initial spectrum:

$$f(x) = \sum_k \left(\frac{G_n}{L}\right)^{1/2} e^{-|k|^n + ikx + i\phi_k},$$

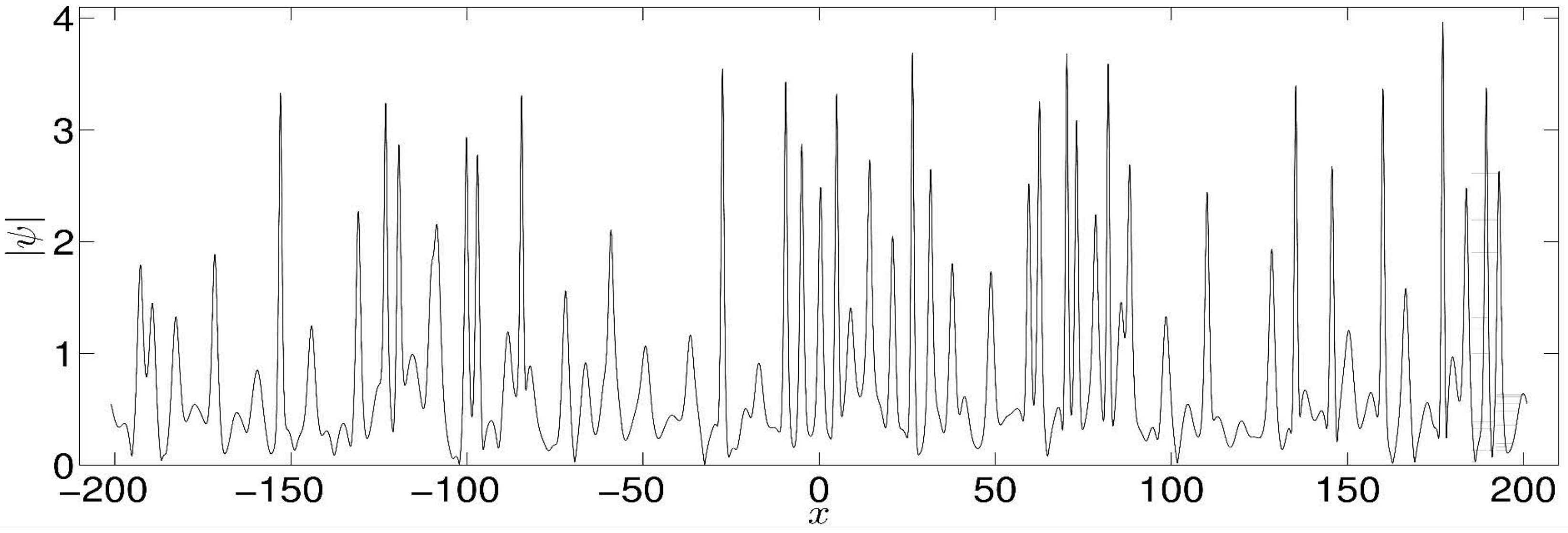
$$\psi_{t=0} = A_0 f(x).$$

Results: dependency on final amplitude A_f.



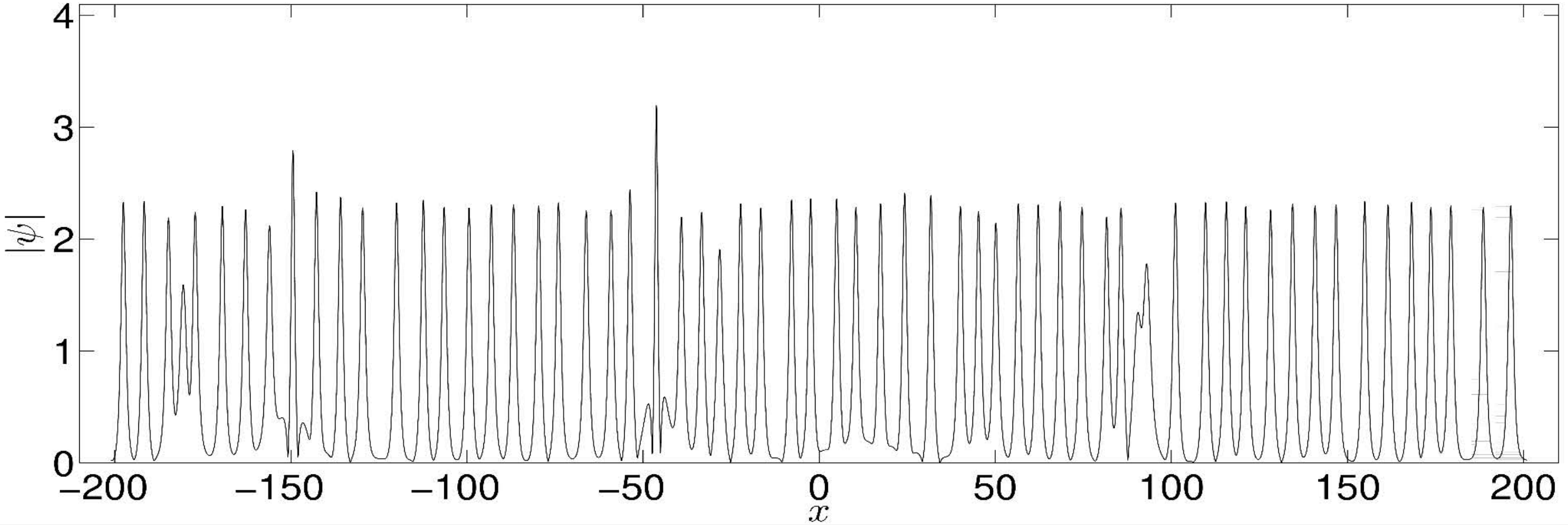
L = 128π, n=2, $A_0 = 10^{-2}$, $p_0 = 10^{-5}$, comparison of different final amplitudes $\mathbf{A_f}$.

Results: wavefield.



L = 128π, n=2, $A_0 = 10^{-2}$, $p_0 = 10^{-5}$, $A_f = 1$.

$$f(x) = \sum_k \left(\frac{G_n}{L}\right)^{1/2} e^{-|k|^n + ikx + i\phi_k}.$$

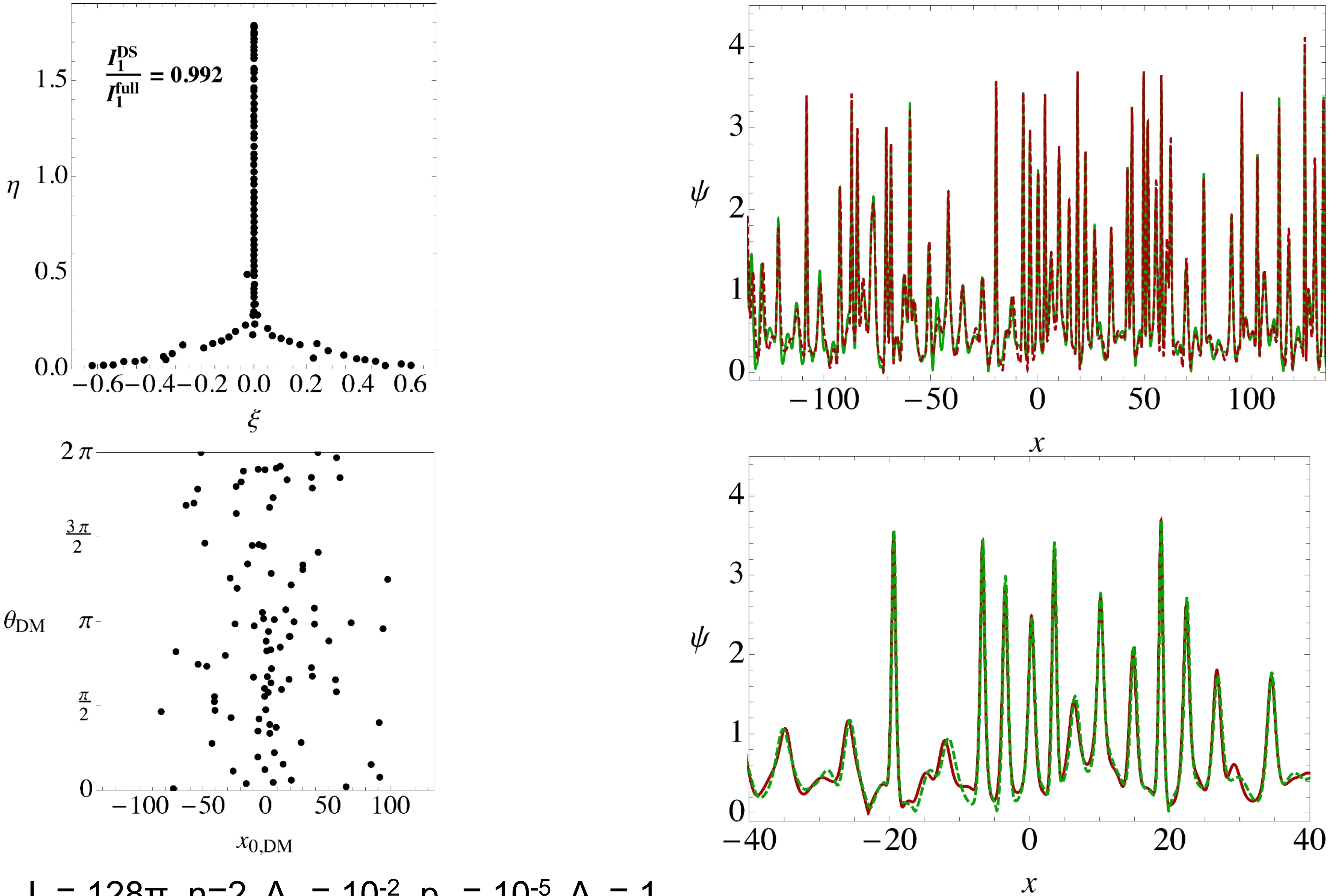


L = 128π, n=32, $A_0 = 10^{-2}$, $p_0 = 10^{-5}$, $A_f = 1$.

$$f(x) = \sum_k \left(\frac{G_n}{L}\right)^{1/2} e^{-|k|^n + ikx + i\phi_k}.$$

Results: IST spectrum.

This work is done by Andrey Gelash and Rustam Mullyadzhanov.



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Results: IST spectrum.

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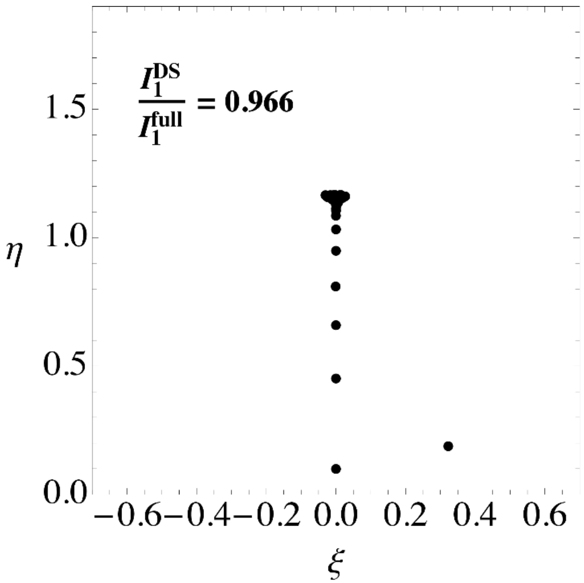
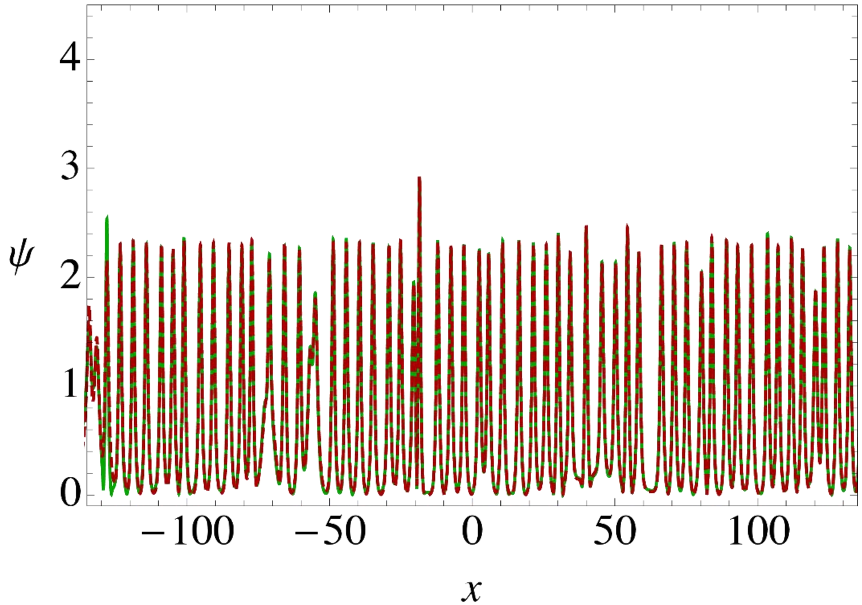
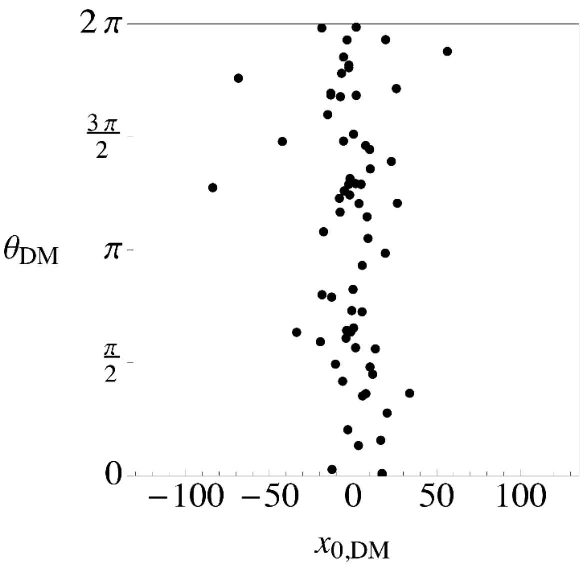
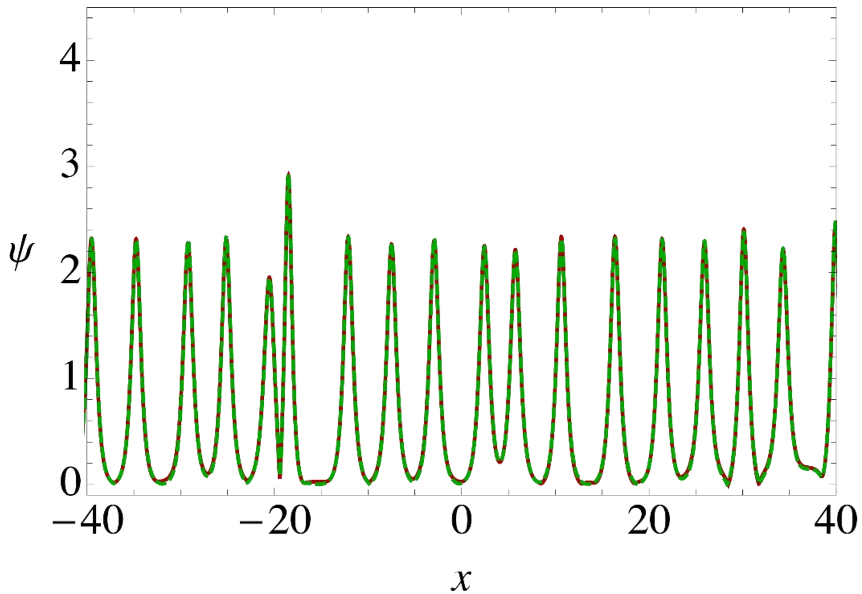


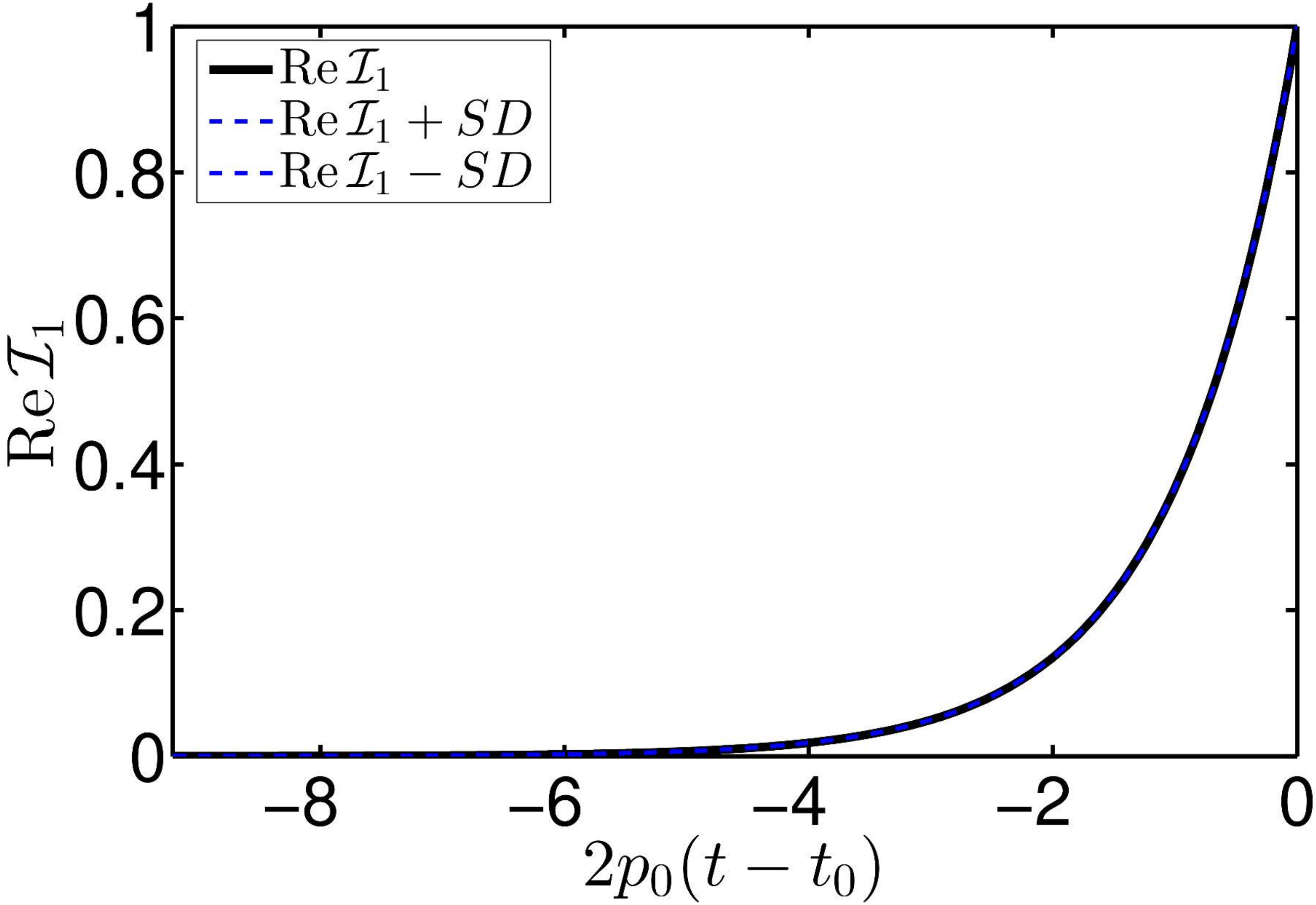






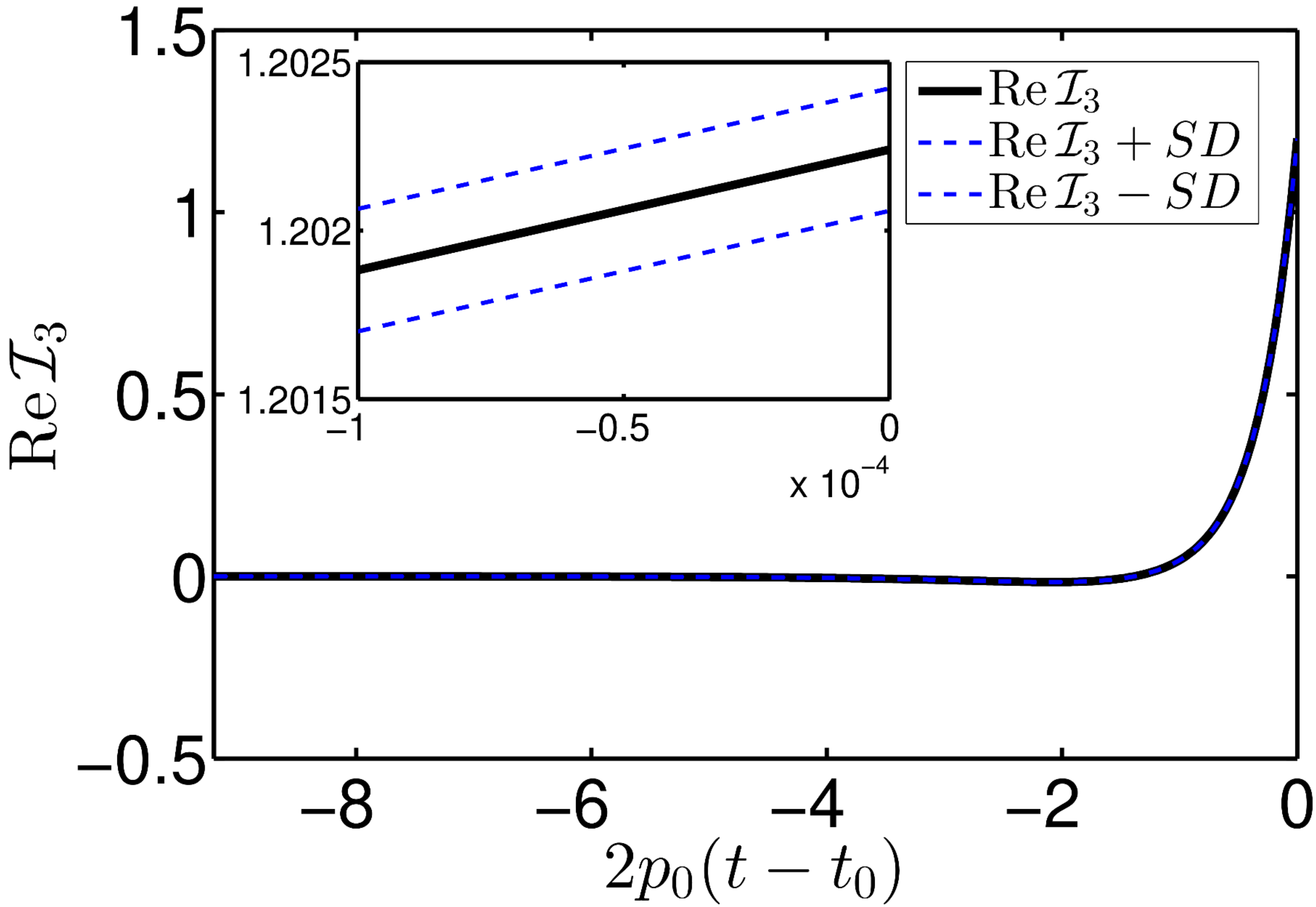
L = 128π, n=32, $A_0 = 10^{-2}$, $p_0 = 10^{-5}$, $A_f = 1$.

Results: integrals of motion during the pumping stage.



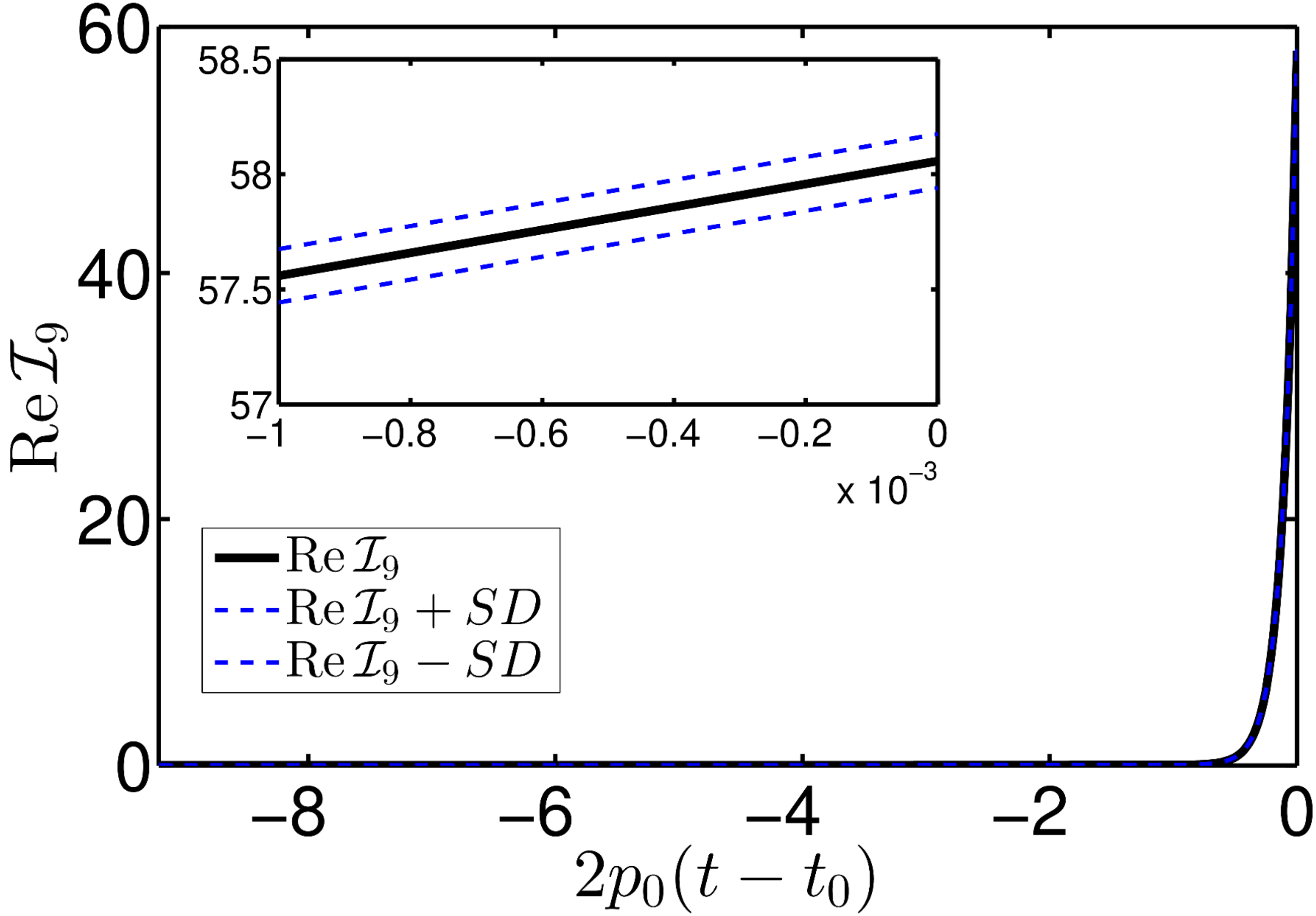
Wave action. L = 128π, n=32, $A_0 = 10^{-2}$, $p_0 = 10^{-5}$, $A_f = 1$.

Results: integrals of motion during the pumping stage.



Energy. L = 128π, n=32, $A_0 = 10^{-2}$, $p_0 = 10^{-5}$, $A_f = 1$.

Results: integrals of motion during the pumping stage.



Ninth integral. L = 128π, n=32, $A_0 = 10^{-2}$, $p_0 = 10^{-5}$, $A_f = 1$.

Conclusions.

We have studied adiabatic and non-adiabatic regimes of turbulence growth within the 1D-NLSE model with linear pumping.

The adiabatic regime is realized when the initial noise and pumping coefficient are small; the smallness of pumping coefficient includes condition on the simulation box (basin length). In the adiabatic regime, the final statistical state doesn't depend on (i) the pumping coefficient, (ii) the amplitude of the initial noise and (iii) the basin length. The final and intermediate states in this regime are stationary states of the integrable turbulence.

Depending on the final average intensity level, we have arrived to weakly and strongly nonlinear states, and the latter states are characterized with a strongly heavy-tailed statistics. The limit is a soliton gas?

The non-adiabatic regime produces states that are characterized by more pronouncedly heavy-tailed statistics.

Implication: in large basins it is very difficult to achieve the adiabatic regime because of the condition $\mathbf{p_0 \ll \Delta k^2}$; the non-adiabatic regime favors appearance of heavy tails in the distribution of intensity (rogue waves).

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Thank you for your attention!