Homogeneous Euler equation. Universality and gradient catastrophes.

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Homogeneous Euler equation.

$$\frac{\partial u_i}{\partial t} + \sum_{k=1}^{n} u_k \frac{\partial u_i}{\partial x_k} = 0 , \qquad i = 1, \dots, n$$

- simplified version of the Navier-Stockes equation ($n=3$) –
– constant pressure, no viscosity dissipation, dispersion etc.
(Landau & Lifshitz, Hydrodynamics, ..).

Even in such most simplified form it arises in various branches of physics from hydrodynamics to cosmology.

Remarkable property – solvable by the method of characteristics or hodograph equations.

1. $x_i = x_{i0} + u_{i0}(x_0)\, t$, $i = 1, \dots, n$ – Y. B. Zeldovich 1970
Shandarin.. 1989
E. Kuznetsov 2003 ....

2. $x_i = u_i t + f_i(u)$, $i = 1, \dots, n$. D. Fairlie, 1993 !

Our goal - slightly generalized HEE:

$$\frac{\partial u_i}{\partial t} + \sum_{k=1}^{n} \lambda_k(u) \frac{\partial u_i}{\partial x_k} = 0, \quad i = 1, \dots, n.$$

Hodograph equations

$$x_i = \lambda_i(u) t + f_i(u), \quad i = 1, \dots, n.$$

$$u_i(x, t=0) = f_i^{-1}(x) \equiv u_{i0}(x)$$

Basic relations

$$\frac{\partial u_i}{\partial x_k} = (M^{-1})_{ik}, \qquad \frac{\partial u_i}{\partial t} = -\sum_{k=1}^{n} (M^{-1})_{ik} \lambda_k(u), \quad i, k = 1, \dots, n$$

where

$$M_{ik} = t \frac{\partial \lambda_i}{\partial u_k} + \frac{\partial f_i}{\partial u_k}, \quad i, k = 1, \dots, n. \qquad \det M \neq 0.$$

M - is the central object:

$$\frac{dM}{dt} + E = 0, \Rightarrow \frac{dV}{dt} + V^2 = 0, \qquad V = M^{-1}.$$

E. Kuznetsov, 2003

Implicit solutions $u_i(x, t) = u_{i0}(\vec{x} - \vec{u} t), \; i = 1, \dots, n.$ $\quad [D_u = D_{u0}]$

I. Universality - constraints to lower dimensions.

1. Diagonal matrix M: $M_{ik} = 0$, $i \neq k$, $i,k = 1,\dots,n$

1a. dimension n: $\frac{\partial \lambda_i}{\partial u_k} = 0$, $\frac{\partial f_i}{\partial u_k} = 0$, $i,k = 1..n$, $i \neq k$

– n decoupled one-dim. BH equations $\frac{\partial u_i}{\partial t} + \lambda_i(u_i)\frac{\partial u_i}{\partial x_i} = 0$, $i = 1,\dots,n$

1.b Constraint $x_1 = x_2 = \dots = x_n = x$.

$$\frac{\partial u_i}{\partial t} + \lambda_i(u_1,\dots,u_n)\frac{\partial u_i}{\partial x} = 0, \quad i = 1,\dots,n$$

$$t = -\frac{f_i - f_k}{\lambda_i - \lambda_k}$$
$i \neq k$.

and

$$\frac{\frac{\partial f_i}{\partial u_k}}{f_i - f_k} = \frac{\frac{\partial \lambda_i}{\partial u_k}}{\lambda_i - \lambda_k} \quad i \neq k, \; i,k = 1,\dots,n.$$

Riemann invariants u_i, characteristic velocities $\lambda_i(u)$

generalized hodograph method S. Tsarev, 1991.

Hodograph equation

$$x = \lambda_i(u)t + f_i(u), \quad i = 1,\dots,n.$$

Geometrical and Hamiltonian implications ...

2. $X_n = 0$, $\lambda_i = u_i$. $(n-1)$-dimensional

Constraints $(M^{-1})_{ni} = (M^{-1})_{in}$, $i = 1, \ldots, n-1$, $(M^{-1})_{nn} = 0$.

HEE is reduced to $(n-1)$-dimensional system

$$\frac{\partial u_i}{\partial t} + \sum_{k=1}^{n-1} u_k \frac{\partial u_i}{\partial x_k} + \frac{\partial v}{\partial x_i} = 0, \quad i = 1, \ldots, n-1$$

$$\frac{\partial v}{\partial t} + \sum_{k=1}^{n-1} u_k \frac{\partial v}{\partial x_k} = 0. \qquad v = \frac{1}{2} u_n^2.$$

For $n = 4$ - adiabatic and isoenthalpic motion at constant temperature

Indeed, for adiabatic motion (see e.g. Landau-Lifshitz)

$$\frac{\partial u_i}{\partial t} + \sum_{k=1}^{3} u_k \frac{\partial u_i}{\partial x_k} + \frac{1}{\rho} \frac{\partial p}{\partial x_i} = 0, \quad i = 1, 2, 3.$$

$$\frac{\partial s}{\partial t} + \sum_{k=1}^{3} u_k \frac{\partial s}{\partial x_k} = 0.$$

Variation of enthalpy $\frac{\partial w}{\partial x_i} = T \frac{\partial s}{\partial x_i} + \frac{1}{\rho} \frac{\partial p}{\partial x_i} \Rightarrow$ for $w = \text{const}$, $T = \text{const}$: $\frac{1}{\rho} \frac{\partial p}{\partial x_i} = -\frac{\partial (Ts)}{\partial x_i}$

$$\Rightarrow \underline{v = -Ts.}$$

3. Polytropic gas in $n-1$ dimensions:

Constraints

$$x_n = 0, \quad x_i = u_i, \quad i = 1, \dots, n$$

and

$$(M^{-1})_{in} = u_n^a (M^{-1})_{ni}, \quad i = 1, \dots, n-1; \quad (M^{-1})_{nn} = \sum_{k=1}^{n-1} (M^{-1})_{kk}.$$

$\Rightarrow$

$$\frac{\partial u_i}{\partial t} + \sum_{k=1}^{n-1} u_k \frac{\partial u_i}{\partial x_k} = -\frac{1}{a+2} \frac{\partial}{\partial x_i} \left(u_n^{a+2}\right), \quad i = 1, \dots, n-1$$

$$\frac{\partial u_n}{\partial t} + \sum_{k=1}^{n-1} \frac{\partial}{\partial x_k} (u_n u_k) = 0.$$

$a = -1$ – shallow water equation, $u_n = h$ – height.

arbitrary a – polytropic gas:

pressure $P = \frac{1}{a+3} \rho^{a+3}$, $\rho = u_n$. – density.

4. 2+1-dim. system $(n=3)$.

$$x_1 = x_2 \equiv x, \quad x_3 = y$$

Constraints $\quad (M^{-1})_{12} = (M^{-1})_{21} = (M^{-1})_{32} = 0.$

3d HEE is reduced to

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}_t + \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}_x + \begin{pmatrix} \lambda_3 & 0 & 0 \\ 0 & \lambda_3 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}_y = 0.$$

5. Constraints

$$x_2 = x_3 = \dots = x_n = 0, \quad \lambda_1 = u_1, \quad \lambda_2 = -1, \quad \lambda_3 = \dots = \lambda_n = 0.$$

and

$$(M^{-1})_{i2} = (M^{-1})_{i+1,1}, \quad i = 1, \dots, n, \quad (M^{-1})_{n2} = 0.$$

$\Rightarrow$ n-component Jordan system $(x = x_1)$

$$\frac{\partial u_i}{\partial t} + u_1 \frac{\partial u_i}{\partial x} + \frac{\partial u_{i+1}}{\partial x} = 0, \qquad i = 1, \dots, n-1,$$

$$\frac{\partial u_n}{\partial t} + u_1 \frac{\partial u_n}{\partial x} = 0.$$

limit $n \to \infty$ Jordan chain

Martinez-Alonso 2002, Shabat.

Pavlov (2003), Kodama, Konopelchenko (2016).

Result:

Certain subclasses of solutions of the homogeneous Euler equation constrained to lower dimensions provide us with the solutions of several hydrodynamic type equations.

Incorporation of dissipation and dispersion.

1. Infinite-dimensional homogeneous Euler equation.
2. Constraint to the one-dimensional Jordan chain:

$$\frac{\partial u_i}{\partial t} + u_1 \frac{\partial u_i}{\partial x} + \frac{\partial u_{i+1}}{\partial x} = 0, \quad i = 1, \ldots, \infty$$

3. Constraint $u_2 = \frac{\partial u_1}{\partial x}$ $\Rightarrow$ $\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + \frac{\partial^2 u_1}{\partial x^2} = 0$, Burgers equation
4. Constraint $u_2 = \frac{\partial^2 u_1}{\partial x^2}$ $\Rightarrow$ $\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + \frac{\partial^3 u_1}{\partial x^3} = 0$ KdV equation

Martinez Alonso Shabat, 2002.
Konopelchenko, Ortenzi, 2018.

II. Blow-up of derivatives, gradient catastrophe.

Homogeneous Euler equation ($x_i = u_i$)
Hodograph equations $\Rightarrow$

$$\frac{\partial u_i}{\partial x_k} = (M^{-1})_{ik} = \frac{\hat{M}_{ik}}{\det M}, \qquad \frac{\partial u_i}{\partial t} = -\sum_{k=1}^{n} (M^{-1})_{ik} u_k = -\frac{1}{\det M} \sum_{k=1}^{n} \hat{M}_{ik} u_k.$$

$i, k = 1 \ldots n.$

Blow-ups $\qquad \det M = 0. \qquad M_{ik} = t\,\delta_{ik} + \frac{\partial f_i}{\partial u_k}, \quad i, k = 1, \ldots, n.$

$\Rightarrow$ blow-up hypersurface in $n+1$ dim. space, $(t, u_1, \ldots, u_n)$

$$t^n + a_{n-1}(u)\, t^{n-1} + \ldots + a_0(u) = 0.$$

real roots.

m real branches

$$t = \bigcup_{i \in S} t_i(u), \qquad t_i(u) = \varphi_i(u), \quad i = 1 \ldots$$

3 general and novel properties.

1. Real roots of the polynomial of the order n with real coefficients

Properties: maximal number m : $= n$.

minimal number m : $= 1$, for $n = 1, 3, 5 \dots$

$= 0$, for $n = 2, 4, 6, \dots$

Hence

1. For $n = 1, 3, 5, \dots$ – any solution of HEE exhibits blow-up.
2. For $n = 2, 4, 6, \dots$ – there are solutions blow-up free.

Particular case of potential flow:

$$u_i = \frac{\partial \varphi}{\partial x_i}, \quad \Rightarrow \quad M_{ik} = \frac{\partial^2 W}{\partial u_i \partial u_k}, \quad i, k = 1, \dots, n \quad \text{since } f_i = \frac{\partial \tilde{W}}{\partial u_i}.$$

M – symmetric matrix with real ~~to~~ elements

$\Rightarrow$ all roots are real $\Rightarrow$ (roots $= -t_i$)

Any potential solution of HEE at any dimension exhibits blow-up (grad. cat. at $t_{min} > 0$).

2. t_{min} for gradient catastrophe

"minimal" branch

$$t_c = \varphi(u)$$

$t_{c\,min}$ corresponds to $\frac{\partial t_c}{\partial u_i} = \frac{\partial \varphi}{\partial u_i} = 0$, $i=1,\dots,n$ + conditions for $\frac{\partial^2 \varphi}{\partial u_i \partial u_k}$

For generic $f_i(u)$ this system of n equations has solution at the point $u_{1c}, u_{2c}, \dots u_{nc}$ $\Rightarrow$

$$t_{c\,min} = \varphi(u_c)$$

So, gradient catastrophe occurs at the point $(u_{1c}, \dots, u_{nc})$ at the time $t_{c\,min} = \varphi(u_c)$ and then expand on the whole blow up hypersurface.

for KP equation Manakov, Santini (2008)

3. Non blow-up subspaces

$$\frac{\partial u_i}{\partial x_k} = \frac{\hat{M}_{ik}}{\det M}, \quad i,k=1..n.$$

blow-up $\det M(t_0) = 0 \Rightarrow$

$$\det M(t=t_0+\varepsilon) = \varepsilon^n + A_{n-1}\varepsilon^{n-1} + \dots + A_1 \varepsilon$$

$$\hat{M}_{ik}(t=t_0+\varepsilon) = \hat{M}_{ik}(t_0) + \varepsilon \hat{\hat{M}}_{ik}(t_0) + \dots$$

$\Rightarrow$

$$\frac{\partial u_i}{\partial x_k} = \frac{\tilde{M}_{ik}(t_0)}{A_1}\frac{1}{\varepsilon} + O(1), \quad \varepsilon \to 0, \quad i,k = 1,\dots,n.$$

So all $\frac{\partial u_i}{\partial x_k} \to \infty$ as $\varepsilon \to 0$.

Principal difference of the multidimensional case from the one-dimensional case ($M = t + \frac{\partial f}{\partial u}$)

$M_{ik}(t_0) \neq 0$, $\tilde{M}_{ik}(t_0) \neq 0$, $i,k = 1,\dots,n$, but $\det M(t_0) = 0$.

Let $r = \operatorname{rank}(M) \leq \operatorname{rank}(\tilde{M}) = \tilde{r}$.

There are $n - \tilde{r}$ real vectors $\vec{R}^{(\alpha)} = (R_1^{(\alpha)}, \dots, R_n^{(\alpha)})$ $\alpha = 1,\dots,n-\tilde{r}$ and $\vec{L}^{(\beta)} = (L_1^{(\beta)}, \dots, L_n^{(\beta)})$, $\beta = 1,\dots,n-\tilde{r}$ such that

$$\sum_{k=1}^{n} \tilde{M}_{ik}(t_0) R_k^{(\alpha)} = 0, \quad \alpha = 1,\dots,n-\tilde{r}, \quad i = 1,\dots,n$$

$$\sum_{i=1}^{n} L_i^{(\beta)} \tilde{M}_{ik}(t_0) = 0, \quad \beta = 1,\dots,n-\tilde{r}, \quad k = 1,\dots,n$$

$$\sum_{\alpha=1}^{n-\tilde{r}} \sum_{k=1}^{n} a_\alpha R_k^{(\alpha)} \frac{\partial u_i}{\partial x_k} \sim O(1), \quad i = 1,\dots,n; \qquad \sum_{\beta=1}^{n-\tilde{r}} \sum_{i=1}^{n} b_\beta L_i^{(\beta)} \frac{\partial u_i}{\partial x_k} \sim O(1), \quad k = 1,\dots,n \qquad a_\alpha, b_\beta \text{ – arbitrary}$$

non blow-up subspaces of dimension $n - \tilde{r}$.

III Examples

1. Two-dimensional case

notation $x_1 = x$, $x_2 = y$, $u_1 = u$, $u_2 = v$, $f_1 = f$, $f_2 = g$.

Two roots.

$$t_\pm(u,v) = \frac{1}{2}\left(-(f_u + g_v) \pm \sqrt{(f_u - g_v)^2 + 4 g_u f_v}\right).$$

a) if $\Delta \equiv (f_u - g_v)^2 + 4 g_u f_v > 0$ - various cases ← always satisfied for potential flows

b) if $\underline{\Delta < 0}$ - no real branches - <u>no blow-ups</u>.

Complex form: $z = x + iy$, $V = u + iv$, $F = f + ig$

2D HEE

$$V_t + V V_z + \bar{V} V_{\bar{z}} = 0$$

hodograph equation

$$z = Vt + F(V, \bar{V})$$

blow-up hypersurface

$$\det M = (F_V + t)(\bar{F}_{\bar{V}} + t) - F_{\bar{V}} \bar{F}_V = 0.$$

Special class of solutions:

$$V_{\bar z} = \mu(z,\bar z,t)\, V_z$$

$\Rightarrow$ HEE

$$V_t + (V + \mu \bar V) V_z = 0.$$

and

$$\mu_t V_z - \mu\left((V+\mu\bar V)V_z\right)_z + \left((V+\mu\bar V)_z\right)_{\bar z} = 0.$$

blow-up surface

$$F_{\bar V}\, \bar F_V (|\mu|^2 - 1) = 0.$$

Particular case:

$\underline{\mu = 0}$, $V_{\bar z} = 0$, $V_t + V V_z = 0$ ←

no blow-up in physical region.

E. Kuznetsov, M. Spector, V. Zakharov. (1994)
E. Karabut, E. Zhuravleva (2014)
N. Zubarev, E. Karabut (2018).

<u>potential flows</u>

$|\mu| < 1$ - quasi-conformal mapping - no blow-up.

blow-up - $|\mu| = 1$ - <u>singularity of quasi-conformal mapping.</u>

Concrete examples.

1. $u_0 = th(x+2y),\ v_0 = th(x+y),$ $f(u) = -arcth(u) + 2arcth(v),\ g(u,v) = arcth(u) - arcth(v)$

$\Rightarrow$ gradient catastrophe at $t_{min} = 1 + \sqrt{2}$.

and $\sqrt{2}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \sim O(1),\ \sqrt{2}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \sim O(1)$.

2. $u_0 = th(x + \varepsilon y),\ v_0 = th(\varepsilon x + y)$ ε - parameter

$\varepsilon = 0$ – two one-dim. BH eqs. - no gradient catastrophe

$1+\varepsilon$ dimension

Gradient catastrophe at $t_{min} = \frac{1}{\varepsilon - 1}$, $(u_c = v_c = 0)$.

3. 3-dim. case

$u_0 = th(x + \varepsilon y),\ v_0 = th(y + \varepsilon z),\ w_0 = u_3 = th(z + \varepsilon x)$.

Blow-up hypersurface

$\left(t + \frac{1}{1-u^2}\right)\left(t + \frac{1}{1-v^2}\right)\left(t + \frac{1}{1-w^2}\right) + \varepsilon^3 t^3 = 0$ $\Rightarrow$ gradient catastrophe $\boxed{t_{c\,min} = -\frac{1}{1+\varepsilon} \quad \varepsilon < -1}$

$u_c = v_c = w_c = 0$.

Papers:

1. B. G. Konopelchenko and G. Ortenzi, On universality of homogeneous Euler equation, J. Phys. A: Math. Theor., 54 (2021) 205701.

2. B. G. Konopelchenko and G. Ortenzi, Homogeneous Euler equation: blow-ups, gradient catastrophes and singularity of mappings, J. Phys. A: Math. Theor., (to be published); arXiv:2109.07309.

Open problems:

1. Fine structure and hierarchy of gradient catastrophes.
2. Gradient catastrophes for reduced equations.
3. Regularization (e.g. parabolic regularization in 1 d).
4. Comparison with the Navier-Stockes equation.

Thank you!