Ismonodromic quantization of the second Painlevé equation by means of autonomous Hamilton systems with two degrees of freedom

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Nonstationary Schrödinger equations corresponding to solutions of the classical Hamilton system

$$(\lambda_k)'_\tau = H'_{\mu_k}, \qquad (\mu_k)'_\tau = -H'_{\lambda_k} \quad (k = 1, 2, ..., n),$$

with the Hamiltonian $H(\tau, \mu_1, \mu_2, ..., \mu_n, \lambda_1, ...\lambda_n)$ have the form

$$\varepsilon \Psi_\tau = H(\tau, -\varepsilon \frac{\partial}{\partial x_1}, ... - \varepsilon \frac{\partial}{\partial x_n}, x_1, ..., x_n)\Psi. \qquad (I)$$

Here $\varepsilon = i\hbar = ih/(2\pi)$ (h – Plank constant).
Example. Any solution to the second Painlevé equation $\lambda'' = 2\lambda^3 + \tau\lambda$ can be represented as a coordinate λ of Hamilton system with Hamiltonian $H = \mu^2/2 - (2\lambda^4/2 + \tau\lambda^2/2)$. Indeed, this system has the form

$$\lambda'_\tau = H'_\mu = \mu, \qquad (\mu)'_\tau = -H'_\lambda = 2\lambda^3 + \tau\lambda.$$

Corresponding Schrödinger equation has the form

$$i\hbar\Psi'_\tau = -\frac{\hbar^2}{2}\Psi''_{xx} + \frac{x^4 + tx^2}{2}\Psi. \qquad (II)$$

All six Painlevé equations compatibility conditions for linear equations of the form (R. Garnier, 1912):

$$W_{xx} = P(\tau, x)W, \qquad W_\tau = B(\tau, x)W_x - \frac{1}{2}B_x(\tau, x)W.(*)$$

For the second Painlevé equation $\lambda'' = 2\lambda^3 + \tau\lambda$ the two corresponding ODE

$$B = \frac{1}{2(x - \lambda)},$$

$$P = (x^4 - \lambda^4) + \tau(x^2 - \lambda^2) + (\lambda')^2 - \frac{\lambda'}{x - \lambda} + \frac{3}{4(x - \lambda)^2}.$$

have simultaneous solutins $W(\tau, x)$. The change $V = \sqrt{(x - \lambda)}W$ give simultaneous solution of two ODE

$$V_{xx} = \frac{V_x}{x-\lambda} + [(x^4 - \lambda^4) + \tau(x^2 - \lambda^2) + (\lambda')^2 - \frac{\lambda'}{x-\lambda}]V,$$

$$V_\tau = \frac{V_x - \lambda' V}{2(x-\lambda)}.$$

They imply that the solution $V(\ tau, x)$ satisfies the identity

$$V_\tau = \frac{V_{xx}}{2} - [\frac{x^2 + \tau x^2}{2} + H(\tau, \lambda(\tau), \lambda'(\tau)]V.$$

Here the function $H(\tau, \lambda(\tau), \lambda'(\tau))$ with $\lambda = \lambda(\tau)$ and $\mu = \lambda'(\tau)$ coincides the above Hamiltonian of the second Painlevé equation (where $\lambda = \lambda(\tau)$ and $\mu = \lambda'(\tau)$). The transformation $\Psi = \exp\left(\int_{\tau_*}^{\tau} H(\nu, \lambda(\nu), \mu(\nu))d\nu\right)V$ transform it to following analog of the Schrödinger equation (II) with $\hbar = -i$.

$$\Psi_\tau = \frac{\Psi_{xx}}{2} - \frac{x^4 + \tau x^2}{2}\Psi.$$

Similar facts are valid for the all six of Painlevé equations. (Suleimanov B.I., 1988).

The second Painlevé equation $u'' = 2u^3 + tu$ is also compatibility condition of two linear ODU (Flashka-Newell pair)

$$\Psi_\zeta = \begin{pmatrix} -i(4\zeta^2 + t + 2u^2) & 4i\zeta u - 2u'_t) \\ -4i\zeta u - 2u'_t & i(4\zeta^2 + t + 2u^2) \end{pmatrix} \Psi, \qquad (III)$$

$$\Psi_t = \begin{pmatrix} -i\zeta & iu \\ -iu & i\zeta \end{pmatrix} \Psi. \qquad (IV)$$

V.Vasov :For each of the six sectors ($j = 1, 2, \ldots, 6$)

$$\Sigma_j = \{\zeta \in \mathbf{C} \,|\, \pi(j-1) < \arg\zeta < \pi(j+1)\,\} \quad (V) \tag{1}$$

there is globally smooth in the variable $\zeta \in \mathbf{C}$ fundamental solution $\Phi_j(t, \zeta)$ of system (3), which at $|\zeta| \to \infty$ in this sector

$$\Phi_j(t,\zeta) = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{j=1}^{\infty} \frac{P_j(t)}{\zeta^j} \right) \exp\left\{ -i(4\zeta^3 + t\zeta) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}. \tag{2}$$

If the function u satisfies the second Painlevé equation $u'' = 2u^3 + tu$, these fundamental solutions $\Phi_j(t, \zeta)$ can be chosen so that they also satisfy the second ODE system IV with independent variable t.

These simultaneous fundamental solutions of the systems are related to each other through the Stokes matrices

$$\Phi_{j+1}(t,\zeta) = \Phi_j(t,\zeta)S_j, \tag{3}$$

where S_j are triangles

$$S_{2j-1} = \begin{pmatrix} 1 & 0 \\ s_{2j-1} & 1 \end{pmatrix}, \qquad S_{2j} = \begin{pmatrix} 1 & s_{2j} \\ 0 & 1 \end{pmatrix} \tag{4}$$

and *and independent from t*. The set of these Stokes matrices constitutes the set of monodromy data. (The independence of this data from the t variable is the basis for the name IDM.)

Fair relationship

$$s_{k+3} = s_k, \qquad s_1 + s_2 + s_3 + s_1 s_2 s_3 = 0.$$

The points of the complex two-dimensional manifold

$$P_2 = \left\{ (s_1, s_2, s_3) \in \mathbf{C}^3 \quad | \quad s_1 + s_2 + s_3 + s_1 s_2 s_3 = 0 \right\}$$

Although in a general situation the solutions of the Painlevé equations cannot be written out explicitly, the IDM allows one to effectively describe the relationship between the asymptotics of each of the Painlevé transcendents as *t to infty* along any ray of the complex plane. In this sense, the solutions of the Painlevé equations are no worse than, say, the solutions of its linear limit, the Airy equation $A''_{tt} = tA$.

But the solutions of the linear IDM systems themselves for the corresponding Painlevé solutions, although they cannot be solved explicitly, with respect to this kind of constraint formulas *is also to a certain extent not worse* than solutions of the classical hypergeometric equation and its various degenerations (as well as solutions of other linear differential equations admitting explicit representations in the form of integrals of the Fourier type).
For example, for any of the solutions of the Painlevé equations, uniquely defined by the points of a two-dimensional manifold, one can describe the asymptotics as $|\zeta| \to \infty$ of simultaneos solution $\Phi_j(t, \zeta)$ of linear systems IDM (III), (IV) along it any ray of the complex plane ζ. This statement obviously follows from the validity for $\Phi_j(t, \zeta)$ in the sector Σ_j of the asymptotics (2) and the relations and form of the Stokes multipliers. To this we can add the fact that for each of the solutions $\Phi_j(t, \zeta)$ asymptotics as $|t| \to \infty$ along any ray of the complex $t-$ plane. It is clear that similar properties are possessed by solutions of linear systems IDM for all Painlevé equations and their various higher isomonodromic analogs.

Matrix

$$W = (\Psi(\eta))^{-1}\Psi(\zeta)\exp\left(t^3/12 - \int_{t_*}^{t}[u'^2 - u^4 - \tau u^2]d\tau\right)$$

satisfies the it scalar equations

$$8W_t = W_{\zeta\zeta} + W_{\eta\eta} + [16(\zeta^4 + \eta^4) + 8t(\zeta^2 + \eta^2)]W,$$

$$W_{\zeta\zeta} - W_{\eta\eta} = 2\frac{W_\zeta + W_\eta}{\zeta - \eta} + [16(\eta^4 - \zeta^4) + 8t(\eta^2 - \zeta^2)]W.$$

All elements of the matrix $W(\zeta, \eta, t)$ are entire functions of all three of their independent variables

These general solutions also satisfy the equation with the coefficients it independent of t

$$4W'_t = \frac{\zeta^2 W''_{\eta\eta} - \eta^2 W''_{\zeta\zeta}}{\zeta^2 - \eta^2} + \frac{(\zeta^2+\eta^2)(W'_\zeta + W'_\eta)}{(\zeta^2-\eta^2)(\zeta-\eta)} - 16\zeta^2\eta^2 W.$$

Changes $r = \zeta + \eta \quad \rho = \zeta - \eta$ reduce it to the equation

$$4W'_t = W''_{rr} + W''_{\rho\rho} + \frac{(r^2+\rho^2)}{r\rho^2}(-\rho W''_{r\rho} + W'_r) - (r^2-\rho^2)^2 W.$$

Now the dilatation $r = \varepsilon^{-1}x$, $\rho = \varepsilon^{-1}y$ and $t = \mp 4\varepsilon\tau$ with $\varepsilon^3 = \mp i\hbar$ give nonstationary Schrödinger equation

$$i\hbar W_\tau = -\hbar^2[W''_{xx} + W''_{yy} + \frac{(x^2+y^2)}{xy^2}(-yW''_{xy} + W'_x)] - (x^2-y^2)^2 W,$$

The equation corresponds, in particular, to the classical Hamilton system with two degrees of freedom

$$(\lambda_k)'_\tau = H'_{\mu_k}, \qquad (\mu_k)'_\tau = -H'_{\lambda_k} \quad (k = 1, 2),$$

with Hamiltonian

$$H = H(q_1, q_2, p_1, p_2) = p_1^2 + p_2^2 - \frac{q_1^2 + q_2^2}{q_1 q_2} p_1 p_2 + -c\frac{p_1}{q_2} - c\frac{p_2}{q_1} - (q_1^2 - q_2^2)^2.$$

The changes

$$Q_1 = -\frac{q_1^2}{2}, \quad Q_2 = \frac{q_2^2 - q_1^2}{4}, \quad P_1 = \frac{p_1}{q_1} + \frac{p_2}{q_2}, \quad P_2 = -\frac{2p_2}{q_2} \quad (5)$$

give solutions of autonomus Hamilton system

$$(Q_1)'_\tau = H'_{P_1} = 4Q_1P_1 + (4Q_1 - 2Q_2)P_2 + c, \qquad (6)$$

$$(Q_2)'_\tau = H'_{P_2} = (4Q_1 - 2Q_2)P_1 + 4(Q_1 - Q_2)P_2, \qquad (7)$$

$$(P_1)'_\tau = -H'_{Q_1} = -2(P_1 + P_2)^2, \qquad (8)$$

$$(P_2)'_\tau = -H'_{Q_2} = 2P_1P_2 + 2P_2^2 - 32Q_2. \qquad (9)$$

with the Hamiltonisn

$$H = 2Q_1P_1^2 + (4Q_1 - 2Q_2)P_2 + c. \qquad (10)$$

The function $J = 2P_1P_2 + P_2^2 + 32(Q_1 - Q_2)$ has the form $J = 32c(\tau - \tau_0)$. (Without loss of generality, we can further assume that $tau_0 = 0$).
The foolowing formulaes are right:

$$32Q_2 = 2P_1P_2 + P_2^2 + 32Q_1 - 32c\tau, \quad (11)$$

$$32Q_1 = -(P_2)'_\tau + P_2^2 + 32c\tau, \quad (12)$$

(γ – arbitrary constant)

$$P_1 = \frac{-(\varphi'_\tau)^2 + \varphi^4}{256c} - \frac{\tau\varphi^2}{2} + \gamma, \quad (13)$$

$$P_2 = \frac{(\varphi'_\tau)^2 - \varphi^4}{256c} + \frac{\tau\varphi^2}{2} - \frac{\varphi}{2} - \gamma, \quad (14)$$

where φ is equation$(\varphi)''_{\tau\tau} = 2\varphi^3 - 128c\tau\varphi$. (If $c = \mp i\hbar/2$, then te change $\varphi = (32)^{1/2}\varepsilon$ transformed it to ODE $u'' = 2u^3 + tu$.)

Let x and y run through the real axes. This means that by the formulas $\zeta = \varepsilon^{-1}Z$ and $\eta = \varepsilon^{-1}Y$, the variables ζ and η are expressed through the variables Z and Y, also running through all valid values. That is, the variables ζ and η for each of the three possible values of *varepsilon* change along the corresponding three straight lines of the complex plane, made up by two diametrically opposite Stokes rays.
Matrix $\Phi^{-1}(t,\eta)\Phi(t,\zeta)$ has form

$$\begin{pmatrix} \Phi_{22}(t,\eta)\Phi_{11}(t,\zeta) - \Phi_{12}(t,\eta)\Phi_{21}(t,\zeta) & \Phi_{22}(,\eta)\Phi_{12}(\zeta) - \Phi_{12}(\eta)\Phi_{22}(\zeta) \\ \Phi_{11}(t,\eta)\Phi_{21}(t,\zeta) - \Phi_{21}(t,\eta)\Phi_{11}(t,\zeta) & \Phi_{11}(\eta)\Phi_{22}(\zeta) - \Phi_{21}(\eta)\Phi_{12}(\zeta) \end{pmatrix} \tag{15}$$

where

$$\begin{pmatrix} \Phi_{11}(t,\eta) & \Phi_{12}(t,\eta) \\ \Phi_{21}(t,\eta) & \Phi_{22}(t,\eta) \end{pmatrix}$$

–fundamental simultaneous solution of ODE of IDM in form Flashka-Newell.