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Transformation of envelope solitons on a bottom step causes extreme events

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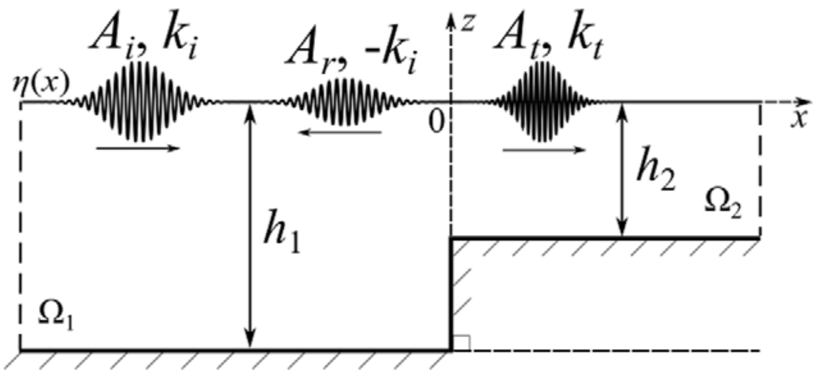
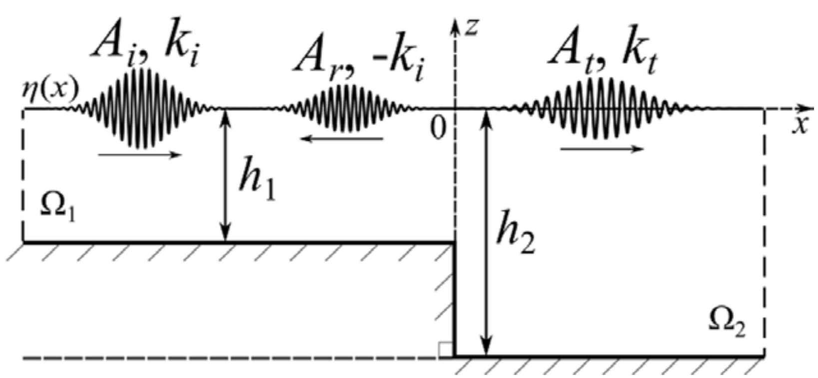
Motivation & Problem setup

We consider the propagation of planar waves on the water surface along the *Ox* axis. The basin consists of two domains with constant depths. The waves experience transformation when they pass the bottom step.

Configuration 1
(towards shallower water)



Configuration 2
(towards deeper water)



The problem of wave transformation on a bottom step has a long story.
The linear problem is investigated since Lamb (1933).
There are some classical results on the nonlinear wave transformation as well.

In the recent time the nonlinear dynamics of waves passing the depth transition is considered in detail in the context of the rogue wave problem:

- Sergeeva et al., NHESS (2011);
- Trulsen et al., PhysFl (2012);
- Zeng & Trulsen, NHESS (2012);
- Trulsen et al., JFM (2020);
- Li et al., JFM (2021);   ……

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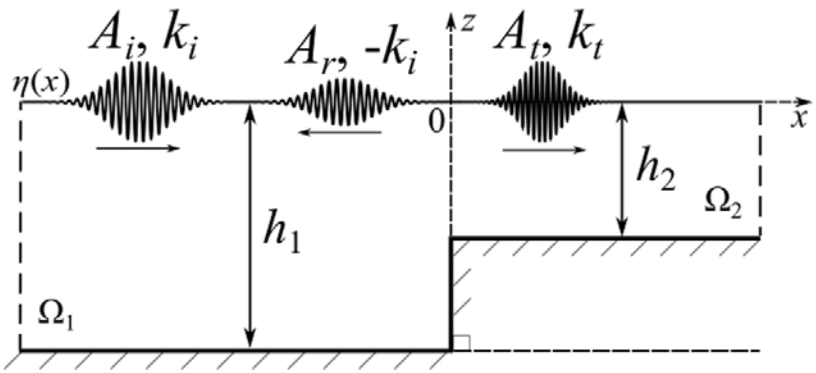
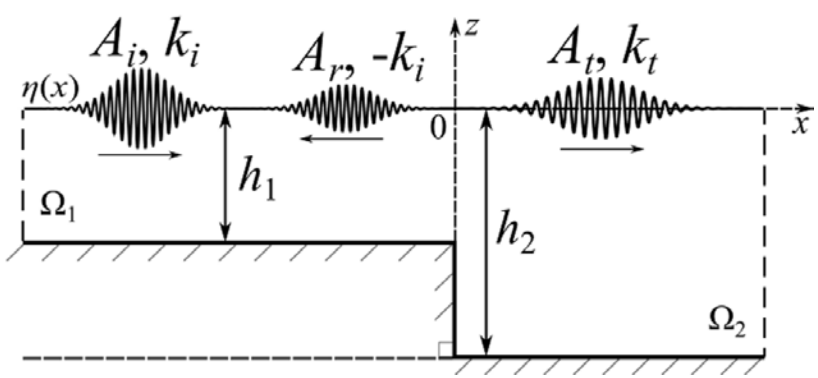


Configuration 2
(towards deeper water)



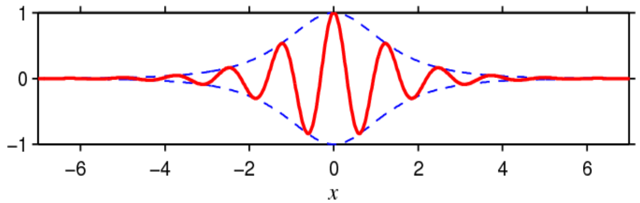
In particular, the surface elevation can have a local maximum of skewness and kurtosis above the shallower part of the shoal (Configuration 1).

The bar-profile or step-profile bottoms were considered in the majority of works, where waves travelled from the deeper to shallower (usually $kh < 1$) water.

We consider the both cases, when the waves travel to either shallower of deeper zones. The most interesting results are obtained for the Configuration 2, when waves enter a deeper zone.

Motivation & Problem setup

We will focus on the situation when waves are modulationally unstable in both domains, $kh_1 > 1.363$, $kh_2 > 1.363$. Then the weakly nonlinear planar theory predicts that envelopes solitons can propagate.



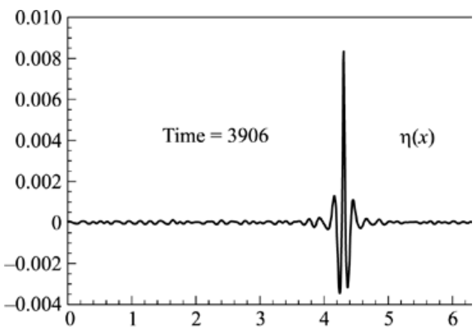
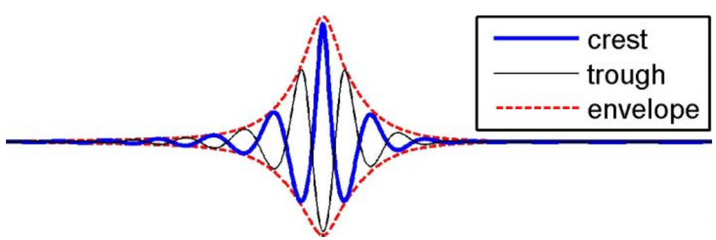
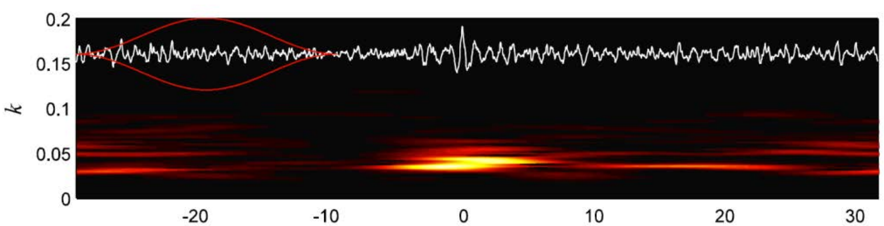


Fig.5. Typical profile of breather





- Stable short nonlinear wave groups with the steepness close to the breaking limit were shown in numerical simulations of Dyachenko & Zakharov (2008).
- Structurally stable hydrodynamic envelope solitons of steep waves were reproduced in laboratory simulations by Slunyaev et al, 2013, 2017.
- Soliton-type short nonlinear wave groups were emerging occasionally from random unidirectional waves with broad spectrum and persisted for more than 200 periods (Slunyaev, 2021).
- The hydrodynamic solitons may be well approximated by the NLS envelope soliton solutions and survive in long-crested wave states (Slunyaev, 2009, 2018, 2021).

Approximate theory

1. The waves in each domain are described by the finite-depth nonlinear Schrodinger equations (NLSEs) for the evolution in space.

$$i\left(\frac{\partial A_1}{\partial x}+\frac{1}{C_1}\frac{\partial A_1}{\partial t}\right)+\beta_1\frac{\partial^2 A_1}{\partial t^2}+\alpha_1|A_1|^2A_1=0 \qquad i\left(\frac{\partial A_2}{\partial x}+\frac{1}{C_2}\frac{\partial A_2}{\partial t}\right)+\beta_2\frac{\partial^2 A_2}{\partial t^2}+\alpha_2|A_2|^2A_2=0$$

$$\eta_j(x,t)=\mathrm{Re}\left(A_j\exp\left(i\omega_j-ik_jx\right)\right)$$ – surface displacement

$$\omega_j=\sqrt{gk_j\sigma_j} \qquad \sigma_j\equiv\tanh(\kappa_j) \qquad \kappa_j\equiv k_jh_j$$ – dimensionless depth / wavenumber

The coefficients of the equations depend on the dimensional depth/wavenumbers κ_j. The group velocity C and the coefficient of dispersion β are always positive, while the coefficient of nonlinearity α changes the sign at $\kappa = kh \approx 1.363$.

Only modulationally unstable conditions will be considered, $\kappa_1 > 1.363$, $\kappa_2 > 1.363$.

$$C(\kappa)=\left.\frac{\partial\omega}{\partial k}\right|_{\omega}=\frac{g}{2\omega}\left(\sigma+\kappa\left(1-\sigma^2\right)\right)$$

$$\beta(\kappa)=\frac{1}{2C^3}\left.\frac{\partial^2\omega}{\partial k^2}\right|_{\omega}=\frac{1}{2C\omega}\left(1-\frac{C_{LW}^2}{C^2}(1-\kappa\sigma)\left(1-\sigma^2\right)\right) \qquad C_p=\frac{\omega}{k} \qquad C_{LW}=\sqrt{gh}$$

$$\alpha(\kappa)=\frac{\omega k^2}{16\sigma^4C}\left(9-10\sigma^2+9\sigma^4-\frac{2\sigma^2C^2}{(C_{LW}^2-C^2)}\left(4\frac{C_p^2}{C^2}+4\frac{C_p}{C}(1-\sigma^2)+\frac{C_{PW}^2}{C^2}(1-\sigma^2)^2\right)\right)$$

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Since the medium parameters do not vary in time, the frequency conserves,

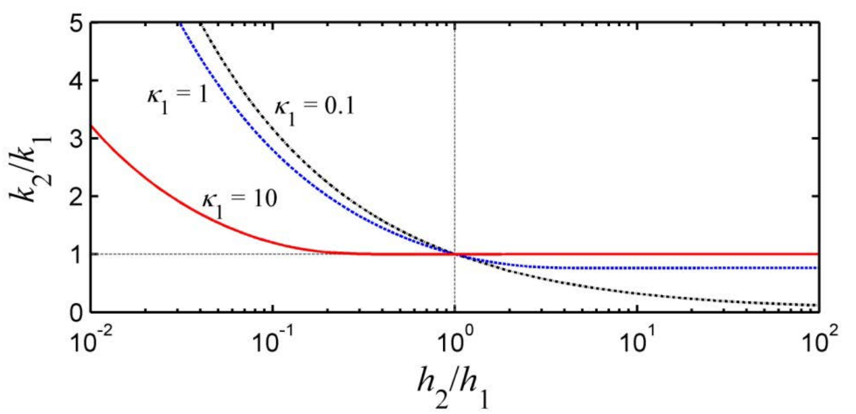
$$\omega_1=\omega_2=\omega.$$

This condition gives us the relation between wavenumbers in the two water domains:

$$k_1\tanh(k_1h_1)=k_2\tanh(k_2h_2).$$

Then it may be shown for the wavenumber as a function of depth $k(h)$ that

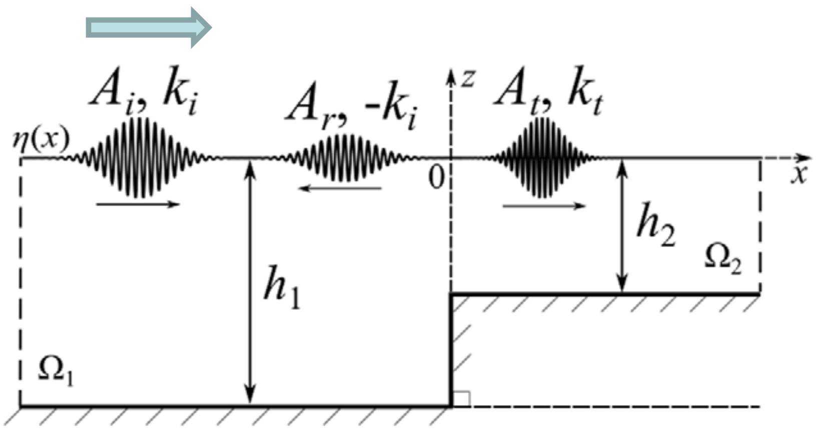
$$\frac{d}{dh}k(h)<0 \qquad \frac{d}{dh}\left[k(h)h\right]>0$$



The wavenumber increases when the wave arrives from a deeper domain.

Approximate theory

2. The solutions of the NLSEs are linked at $x = 0$ using the linear solution suggested in Giniyatullin et al. (2014), Kurkin et al. (2015). Reflected waves will not be considered.



Giniyatullin et al. (2014):

$$T = \frac{A_t}{A_i} = \frac{2C_1}{C_1 + C_2} \qquad R = \frac{A_r}{A_i} = \frac{|C_1 - C_2|}{C_1 + C_2}$$

(Note that in general $T^2 + R^2 \neq 1$ since the coefficients relate the wave amplitudes, not energies)

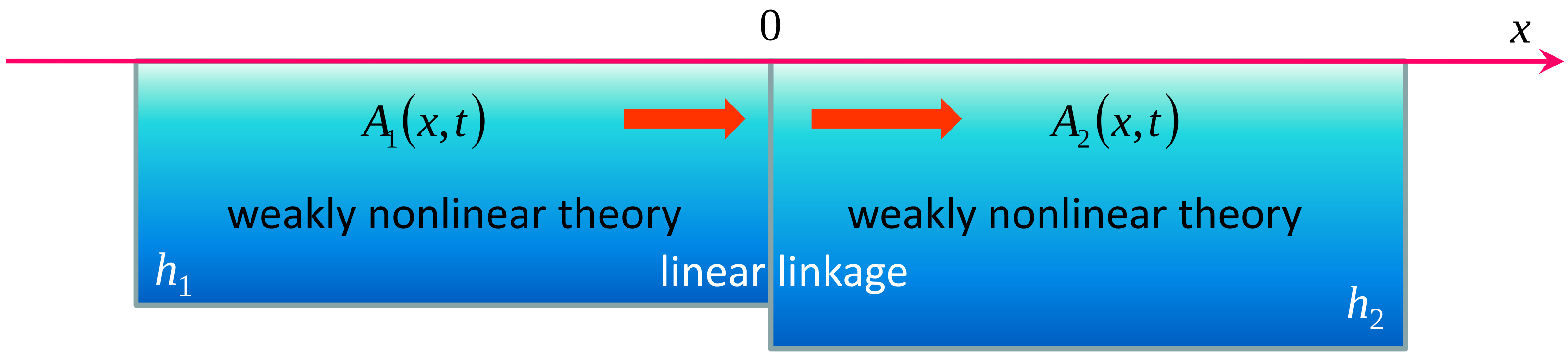
Lamb's formulas for the long wave limit ($C=(gh)^{1/2}$):

$$T = \frac{2}{1 + \sqrt{h_2/h_1}} \qquad R = \frac{1 - \sqrt{h_2/h_1}}{1 + \sqrt{h_2/h_1}}$$

Miles (1967) showed that the transformation coefficients of surface waves can be evaluated with the 95% accuracy by neglecting all the evanescent modes.

Using the numerical simulation of the primitive equations of hydrodynamics, the approximate formulas were shown to serve well in the situation of linear uniform waves [Kurkin et al., 2015].

Approximate theory



$$\boxed{i\left(\frac{\partial A_1}{\partial x}+\frac{1}{C_1}\frac{\partial A_1}{\partial t}\right)+\beta_1\frac{\partial^2 A_1}{\partial t^2}+\alpha_1|A_1|^2A_1=0}$$

$$i\left(\frac{\partial A_2}{\partial x}+\frac{1}{C_2}\frac{\partial A_2}{\partial t}\right)+\beta_2\frac{\partial^2 A_2}{\partial t^2}+\alpha_2|A_2|^2A_2=0$$

The linking condition with the linear transmission coeff.: $A_2(x=0,t)=TA_1(x=0,t)$

After the change of variables $\tilde{A}_2(\tilde{x},t)=S\,A_2(x,t)$ $\quad S=\sqrt{\dfrac{\alpha_2\beta_1}{\alpha_1\beta_2}}$ $\quad \tilde{x}=x\dfrac{\beta_2}{\beta_1}$

The NLSE in the second domain differs from the NLSE in the first domain only in the coefficient of the advection:

$$\boxed{i\left(\frac{\partial \tilde{A}_2}{\partial \tilde{x}}+\frac{1}{\tilde{C}_2}\frac{\partial \tilde{A}_2}{\partial t}\right)+\beta_1\frac{\partial^2 \tilde{A}_2}{\partial t^2}+\alpha_1|\tilde{A}_2|^2\tilde{A}_2=0}\qquad \tilde{C}_2=C_2\frac{\beta_2}{\beta_1}$$

The 'initial' condition at *x* = 0 is the wave from the side of the first domain with altered amplitude,

$$\tilde{A}_2(\tilde{x}=0,t)=ST\cdot A_1(x=0,t)$$

Transformation of an envelope solition

If the incident wave has the form of the envelope soliton, then

$$A_1(x,t) = a\frac{\exp\left(\frac{i}{2}a^2\alpha_1 x\right)}{\cosh\left(a\sqrt{\frac{\alpha_1}{2\beta_1}}\left(t-\frac{x}{C_1}\right)\right)}$$

$$\tilde{A}_2(x=0,t) = \frac{\mu a}{\cosh\left(\sqrt{\frac{\alpha_1}{2\beta_1}}at\right)}, \quad \mu := ST$$

The soliton content of this "initial condition" is known since Satsuma & Yajima (1974). *N* solitons may emerge from the pulse with amplitudes as follows:

$$\tilde{a}_n = 2a\left(\mu - n + \frac{1}{2}\right), \quad n = 1,2,\ldots,N \qquad N = E\left(\mu + \frac{1}{2}\right)$$

$E(\cdot)$ stands for integer part of the argument

In the original (non-scaled) coordinates the solitonic part of the solution reads

$$a_n = 2\frac{a}{S}\left(\mu - n + \frac{1}{2}\right), \quad n = 1,2,\ldots,N \qquad N = E\left(\mu + \frac{1}{2}\right) \qquad S = \sqrt{\frac{\alpha_2\beta_1}{\alpha_1\beta_2}}$$

In the limit $S \to \infty$ (when $\kappa \to 1.363$) the number of solitons N goes to infinite, but their amplitudes in the second domain are limited, $a_n < 2Ta$.

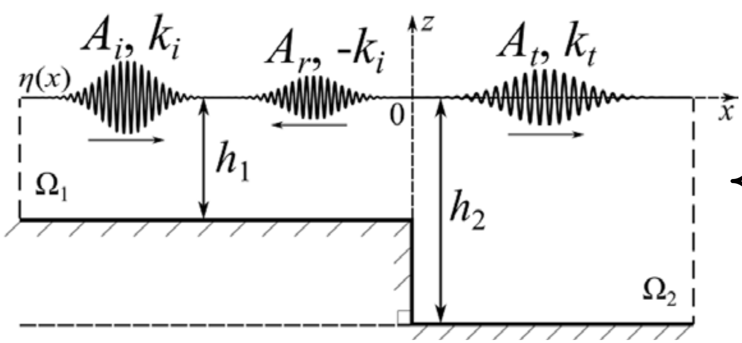
It can be shown that if $h_2 > h_1$, then $S > 1$ and $T > 1$ (the soliton amplitude grows).
In the opposite case $h_2 < h_1$ one has $S < 1$ and $T < 1$ (the soliton amplitude decreases).

(**NB:** These are valid for $k_1h_1 > 1.363$ and $k_2h_2 > 1.363$ only!)

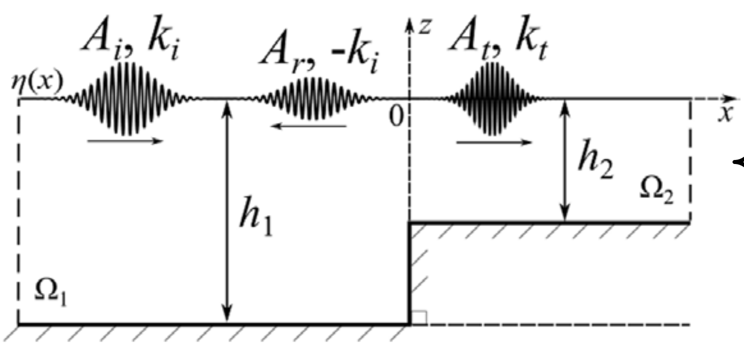
The linear transmission coefficient T

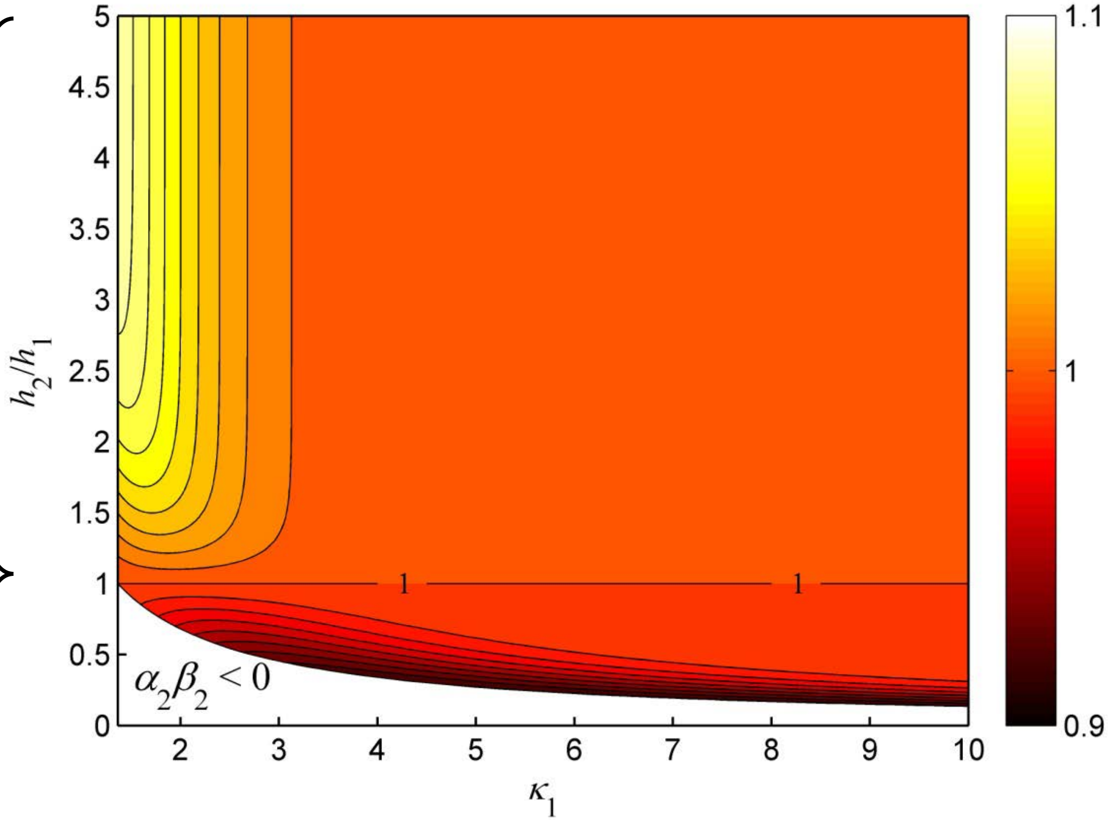
Under the conditions $k_1 h_1 > 1.363$ and $k_2 h_2 > 1.363$ the transmission coefficient T is greater than one if the soliton travels to deeper water. It is less than one in the opposite case. In all situations the wave amplitude alteration is within about 10%. The main contribution to $\mu = ST$ is by S.

Configuration 1



Configuration 2

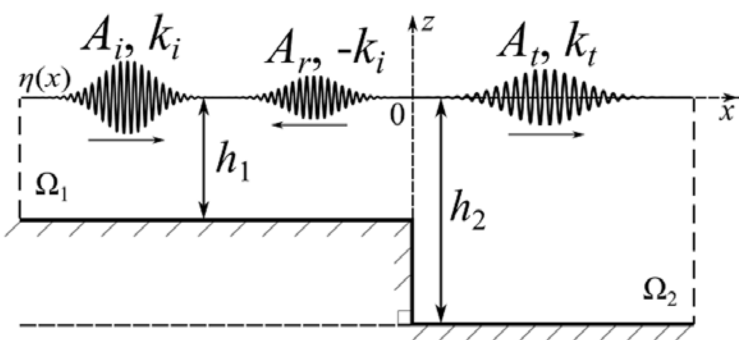




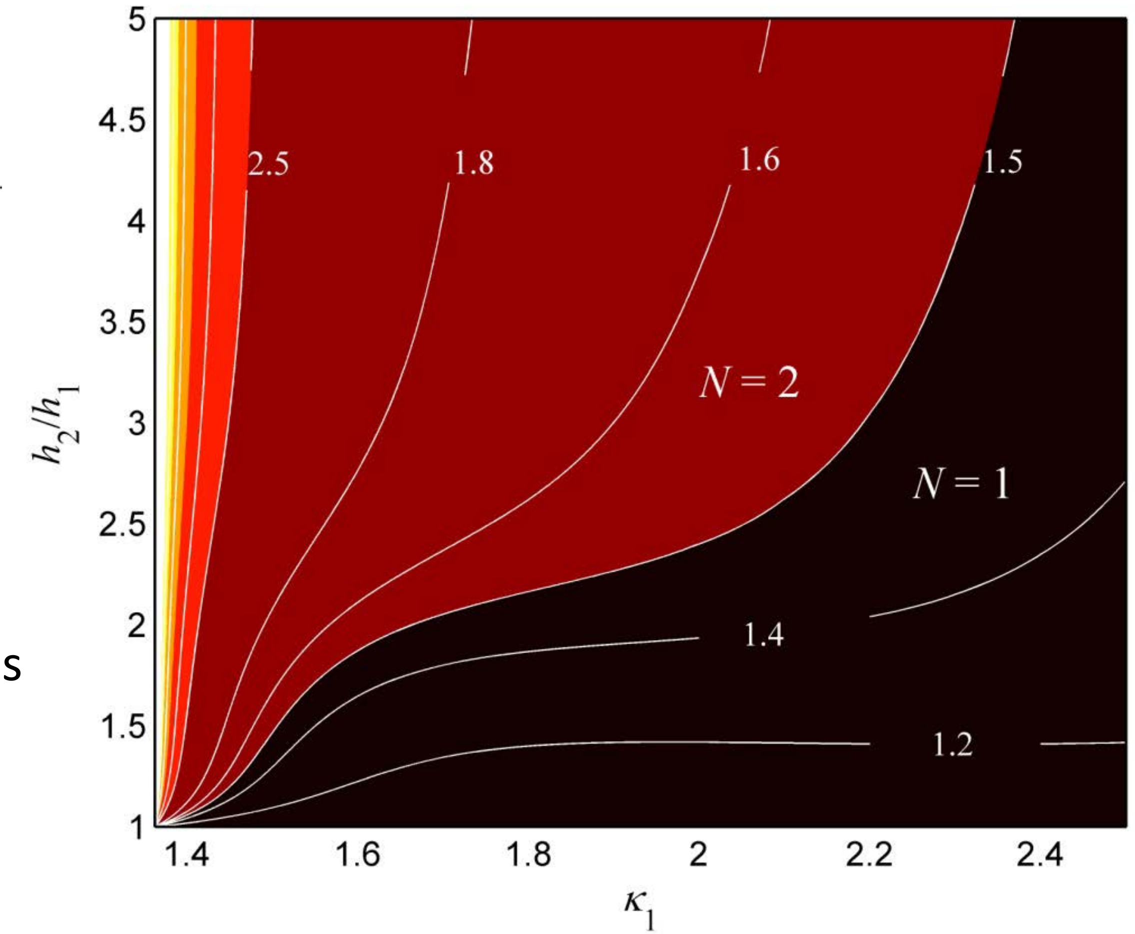
The full transmission coefficient $\mu = ST$

More than one soliton may emerge after the bottom step in the case of a bottom drop. The soliton number N may be unrestrictedly large when $k_1 h_1 \to 1.363$

Configuration 1



If the second domain is infinitely deep, the second solition emerges when $k_1 h_1 < 2.9$. Three solitions appear when $k_1 h_1 < 1.8$.



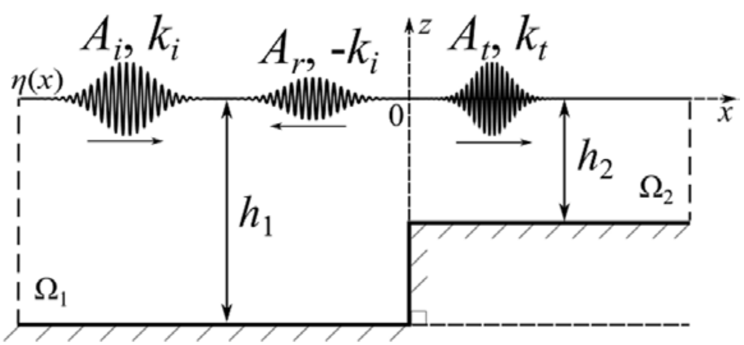
The full transmission coefficient $\mu = ST$

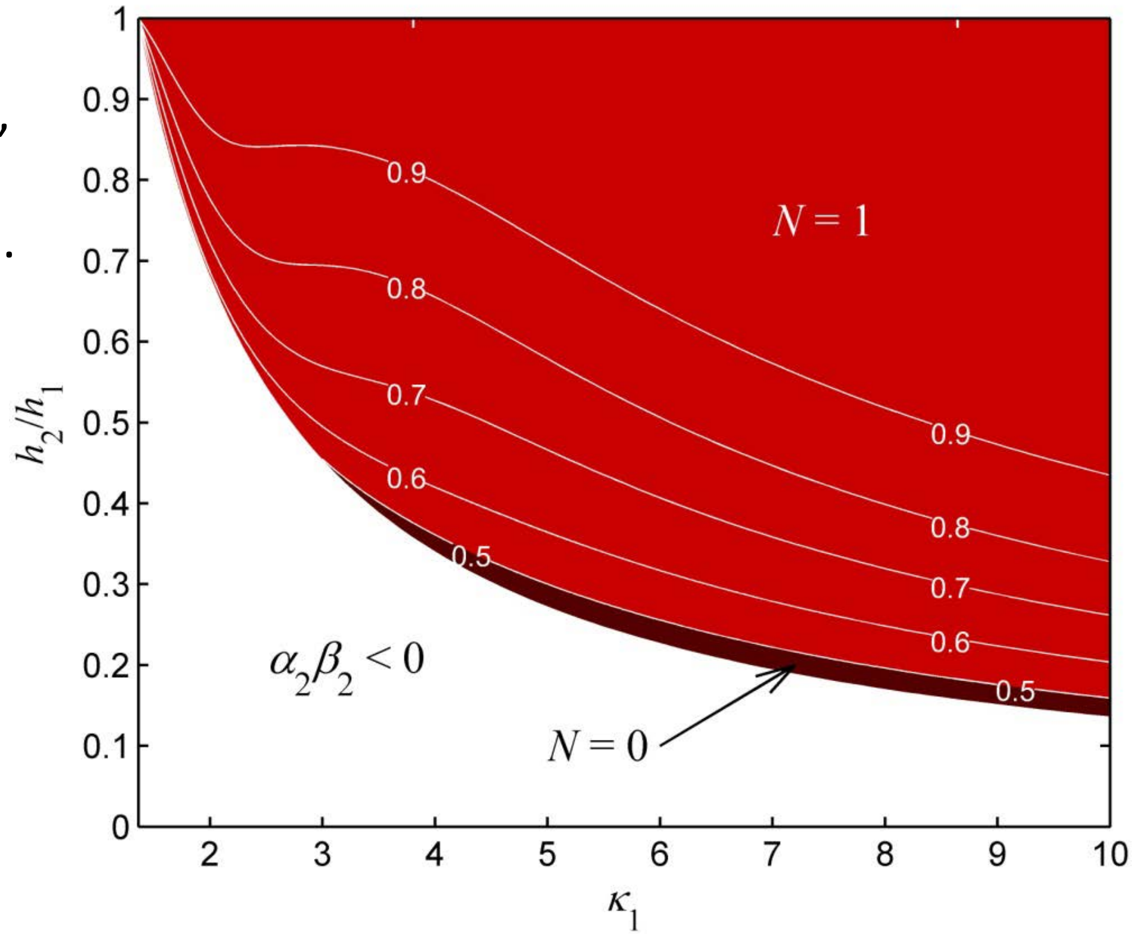
New solitions do not emerge when the envelope soliton travels from deep to shallower water. The solition amplitude diminishes.

If $h_2 << h_1$ or $k_1 h_1$ is too small, then the solliton disintegrates when passes through.

If the first domain is infinitely deep, the soliton completely disappears after the transition when $k_2 h_2 < 2.1$.

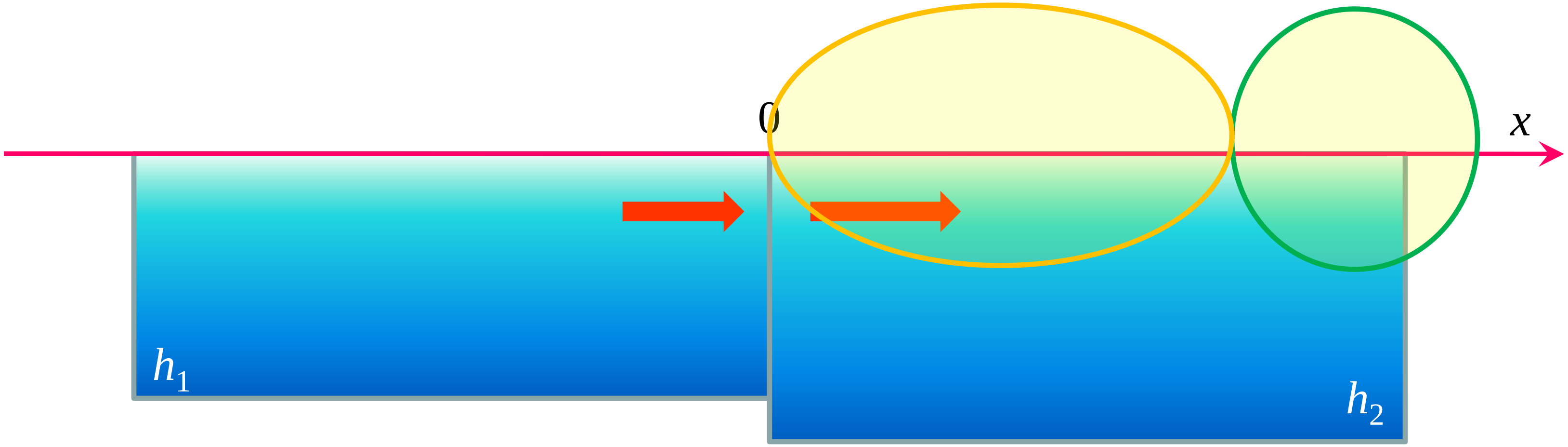
Configuration 2





Asymptotic solution

The obtained solution describes the solitonic part of the transmitted waves when the quasi-linear waves have spread and the solitons are isolated (the asymptotic solution).



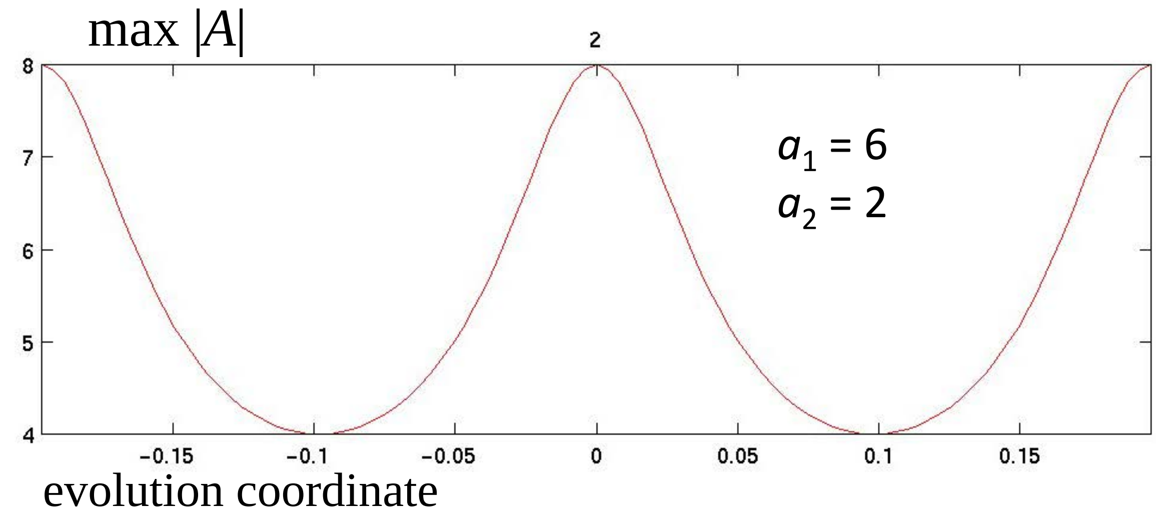
However, all the soltions which emerge in the second domain travel with the same velocity, hence they form bound states of solitons (also known as bi-solitons when $N = 2$) [Satsuma & Yajima, 1974; Peregrine, 1983].

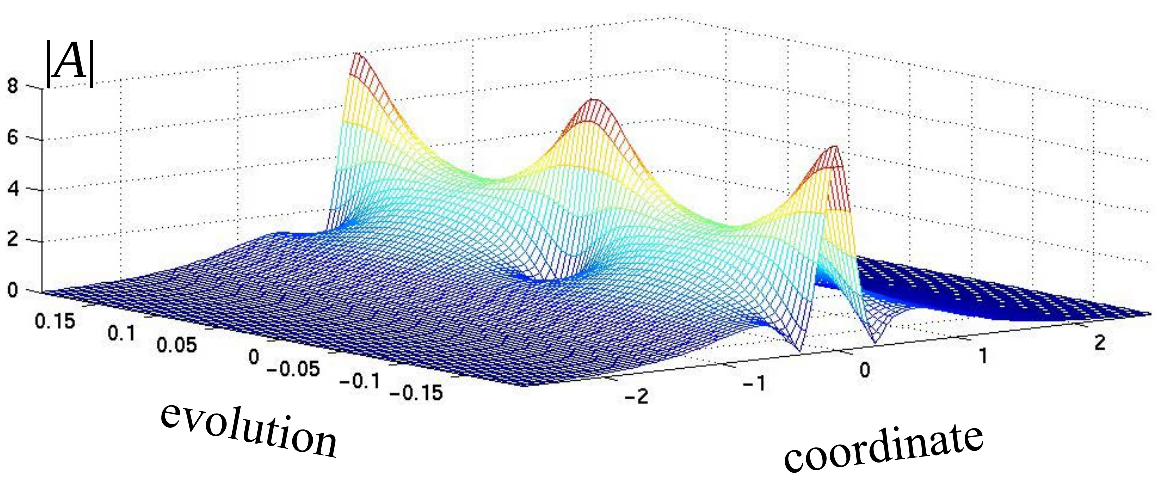
Therefore if $N > 1$, then the solution in the second domain represents a complicated wave pattern, which can produce extreme wave events and hence desires a quantitative description. It remains complicated even far from the bottom step, since the solitons travel with identical speeds and hence will never decouple.

An example of a bi-soliton

A bi-soliton consists of two beating envelope solitions with amplitudes a_1 and a_2, $a_1 > a_2 > 0$. According to the exact solution of the NLSE, the maximum amplitude of a bi-soliton oscillates with the spatial period

$$L_b = \frac{\pi}{4\left(a_1^2 - a_2^2\right)}$$



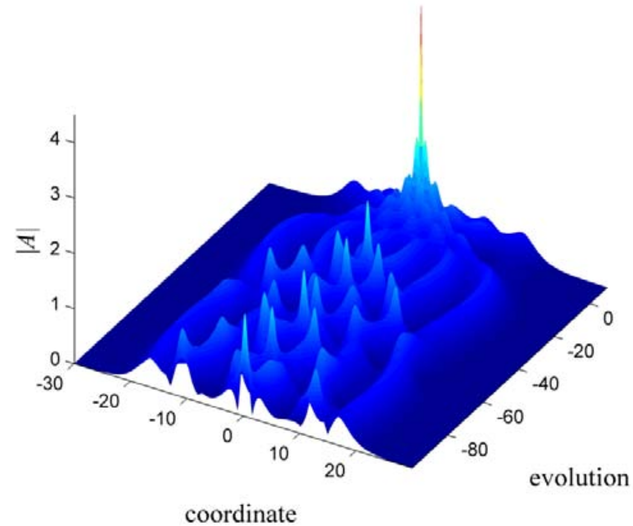


If the soliton amplitudes a_2 and a_1 are not too much similar ($a_2/a_1 <$ 0.38), then the extremes are confined between $a_1 - a_2$ and $a_1 + a_2$.

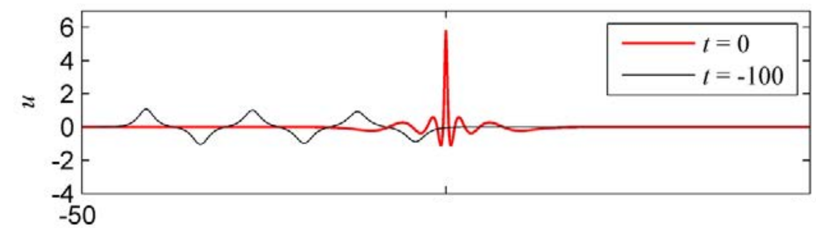
This dynamics resembles much the classic effect of beating between two harmonic waves.

Interactions between many solitons

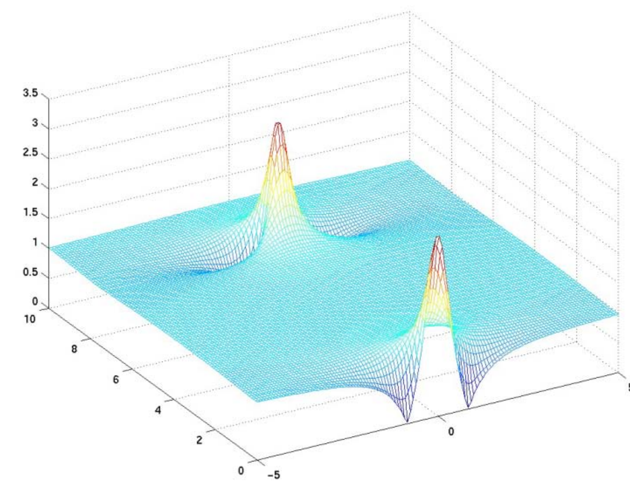
❶ When N envelope solitons interact within the NLSE, the maximum wave amplitude is just the sum of the soliton amplitudes: $|a_1| + |a_2| + \ldots + |a_N|$ [Akhmediev, & Mitzkevich, 1991; Sun, 2016; Slunyaev & Pelinovsky, 2016].



❷ When N solitons of the modified Korteweg – de Vries equation of the focusing type interact, the wave amplitude in the focusing point is $a_1 - a_2 + a_3 - a_4 + \ldots$, where a_j may be either positive or negative, $a_j > a_{j+1}$ [Slunyaev & Pelinovsky, 2016; Slunyaev, 2019]. Depending on the choice of signs of a_j, constructive ($|a_1|+|a_2|+\ldots$) or destructive ($|a_1|-|a_2|+|a_3|-\ldots$) interference can occur.



❸ One Kuznetsov breather on the background plane wave of the amplitude a_{pw} may be interpreted as an envelope soliton with intrinsic amplitude a. The maximum amplitude of the breather solution is confined within $|a - a_{pw}|$ and $|a + a_{pw}|$ [Slunyaev, 2006]. A collision of N Peregrine breathers leads to the maximum amplitude $Na + a_{pw}$ [Wang et al, 2017].



Conjecture on the principle dynamics

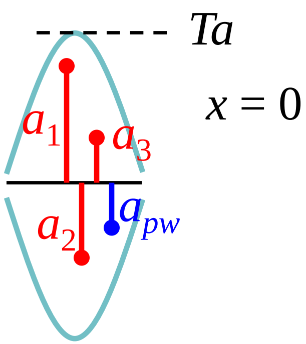
At the boundary of the second domain, $x = 0$, the wave has a pulse-like form (sech shape) with the amplitude Ta.

It contains N envelope solitons with amplitudes a_1, a_2, …, a_N, where $a_j > a_{j+1} > 0$.

1) The wave amplitude consists of a solitonic part a_{sol} and a residual quasi-linear a_{pw} (plane wave) part, $a_{pw} = |a_{sol} - Ta| > 0$.

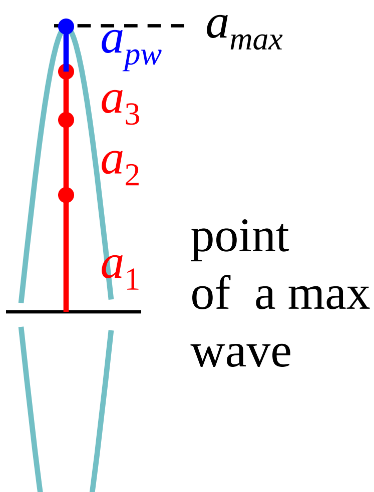
2) At $x = 0$ the solitons are "packed", so that the solitonic part is in the state of a destructive interference,

$$a_{sol} = a_1 - a_2 + a_3 - \ldots + (-1)^{N+1} a_N \,.$$



3) In the course of the evolution of the bound solitons the maximum attainable amplitude is produced by a constructive interference of all the constituents:

$$\boxed{a_{\max} = a_1 + a_2 + \ldots + a_N + a_{pw}}$$

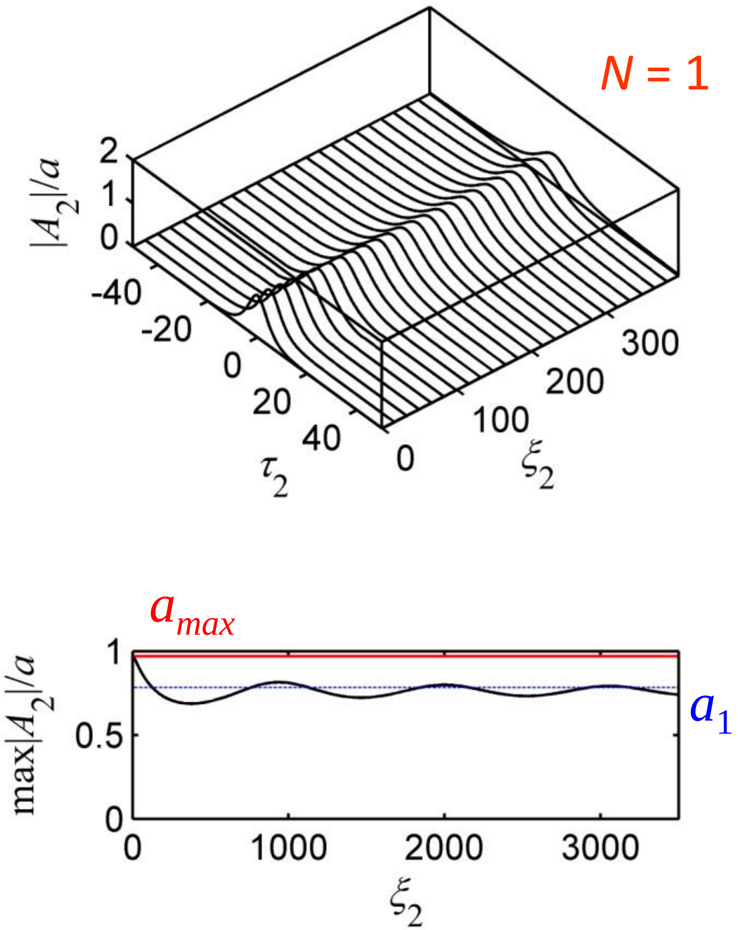


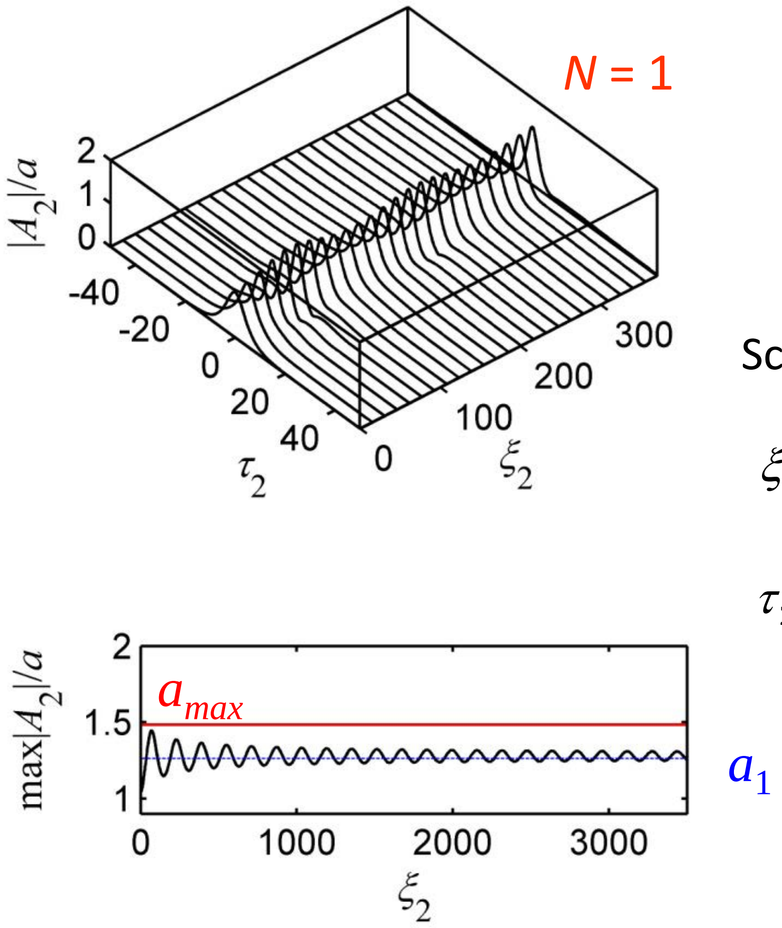
It is a conjecture which is correct in the case of a linear dynamics and does not contradict the known key effects of the soliton dynamics. As a result, we have the analytic formula in hands which estimates the maximum wave amplitude a_{max} which can occur after the depth change.

Numerical simulations of the NLSE

Cases: $k_1h_1 = 2$, $h_2/h_1 = 0.8$ $\quad$ $k_1h_1 = 2$, $h_2/h_1 = 1.5$

$a_1/a \approx 0.78$, $a_{pw}/a \approx 0.09$ $\quad$ $a_1/a \approx 1.26$, $a_{pw}/a \approx 0.22$





Scaled variables:

$$\xi_2 = \frac{k_2}{2\pi}x$$

$$\tau_2 = \frac{\omega_2}{2\pi}\left(t - \frac{x}{C_2}\right)$$

Numerical simulations of the NLSE

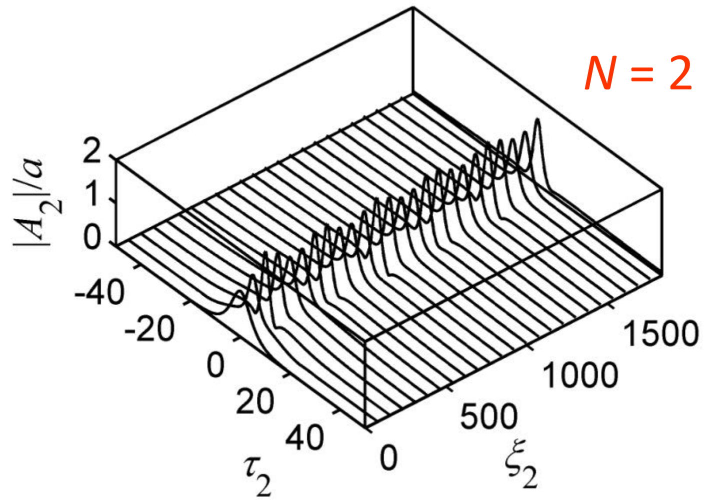
Cases: $k_1h_1 = 2,\ h_2/h_1 = 2.5$

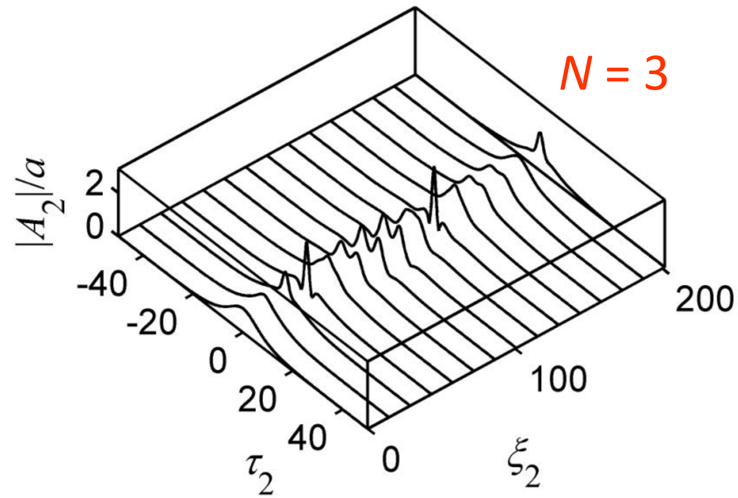
$a_1/a \approx 1.44,\ a_2/a \approx 0.12,\ a_{pw}/a \approx 0.26$

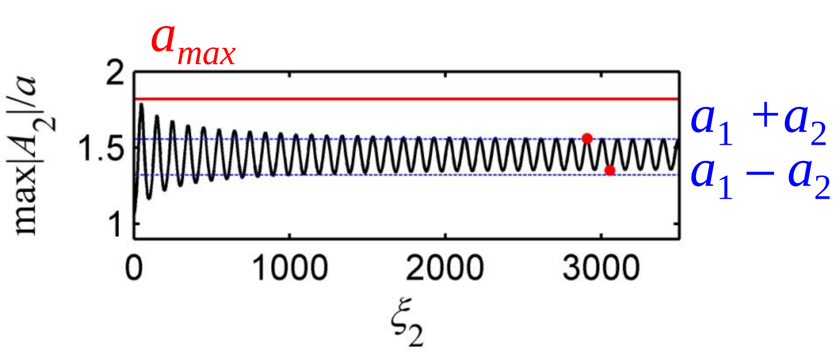
$k_1h_1 = 1.5,\ h_2/h_1 = 13$

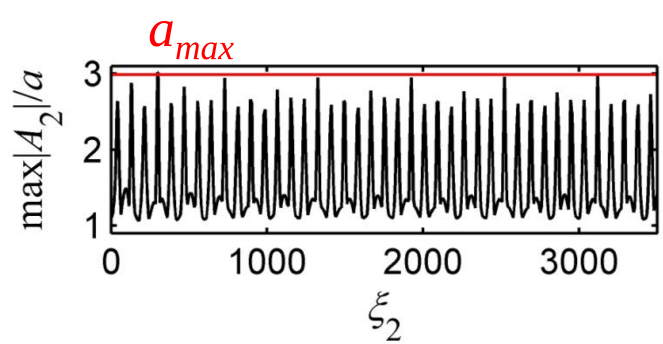
$a_1/a \approx 1.76,\ a_2/a \approx 0.95,\ a_3/a \approx 0.15,$

$a_{pw}/a \approx 0.13$

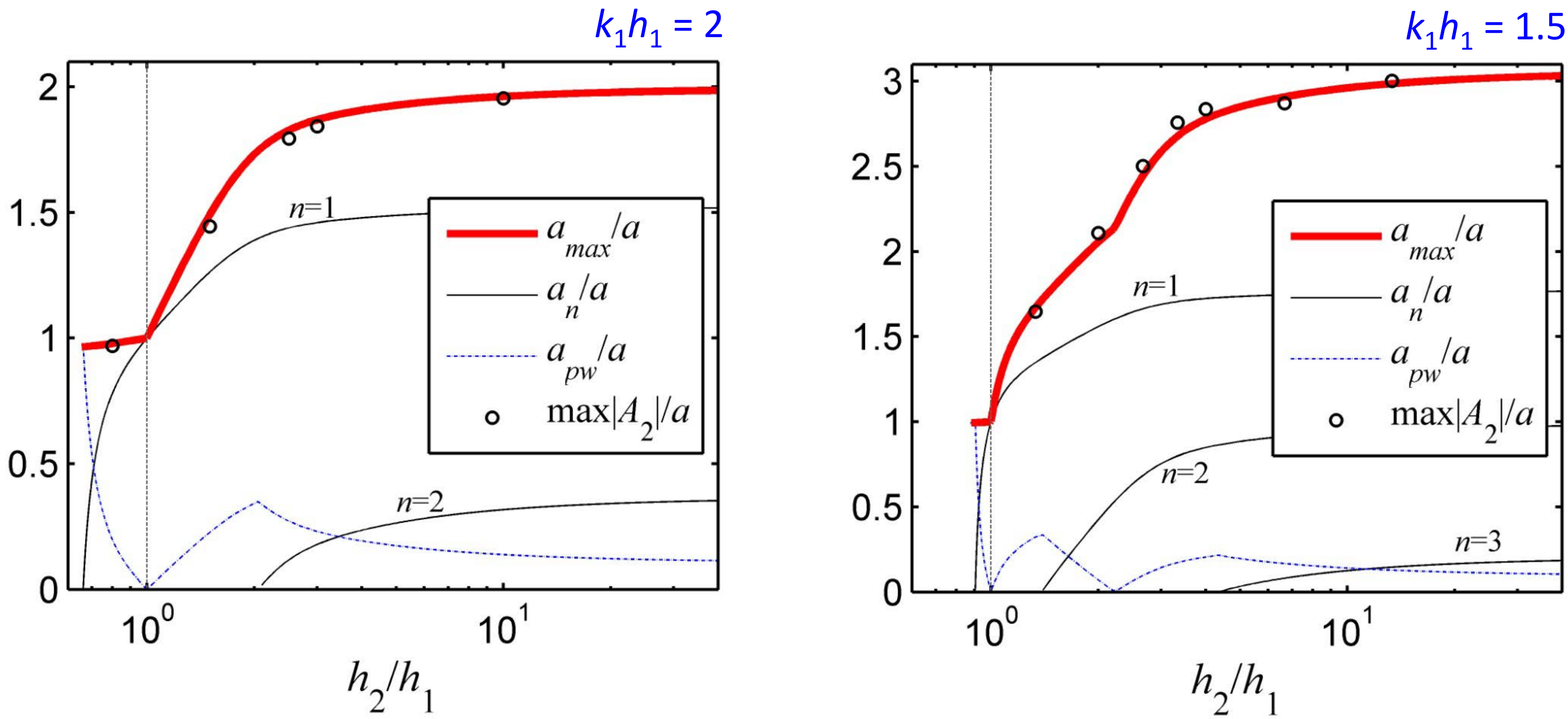








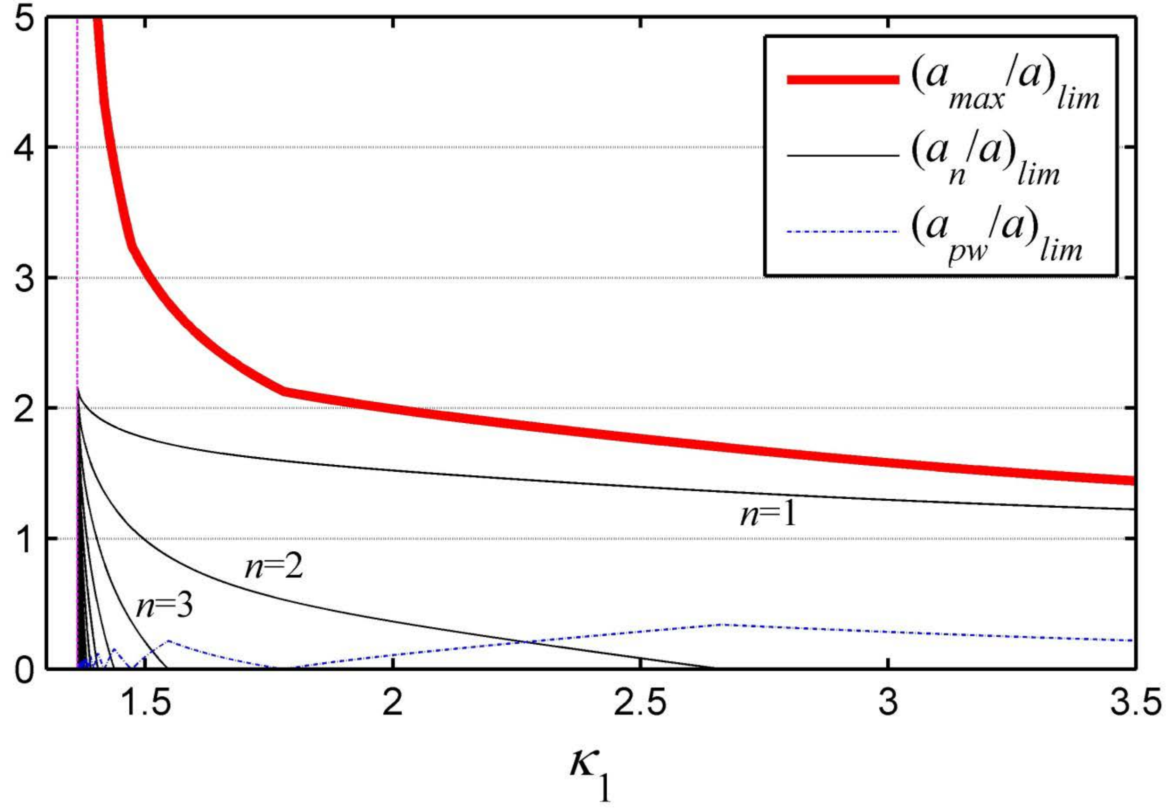
Validation of the suggested a_{max}



The direct numerical simulation of the NLSE (circles) confirms validity of the suggested formula for a_{max} even though the curve shape is complicated.

The role of the residual quasi-linear waves may be significant. They spread so slowly, that the extreme in-phase superposition with solitons does occur.

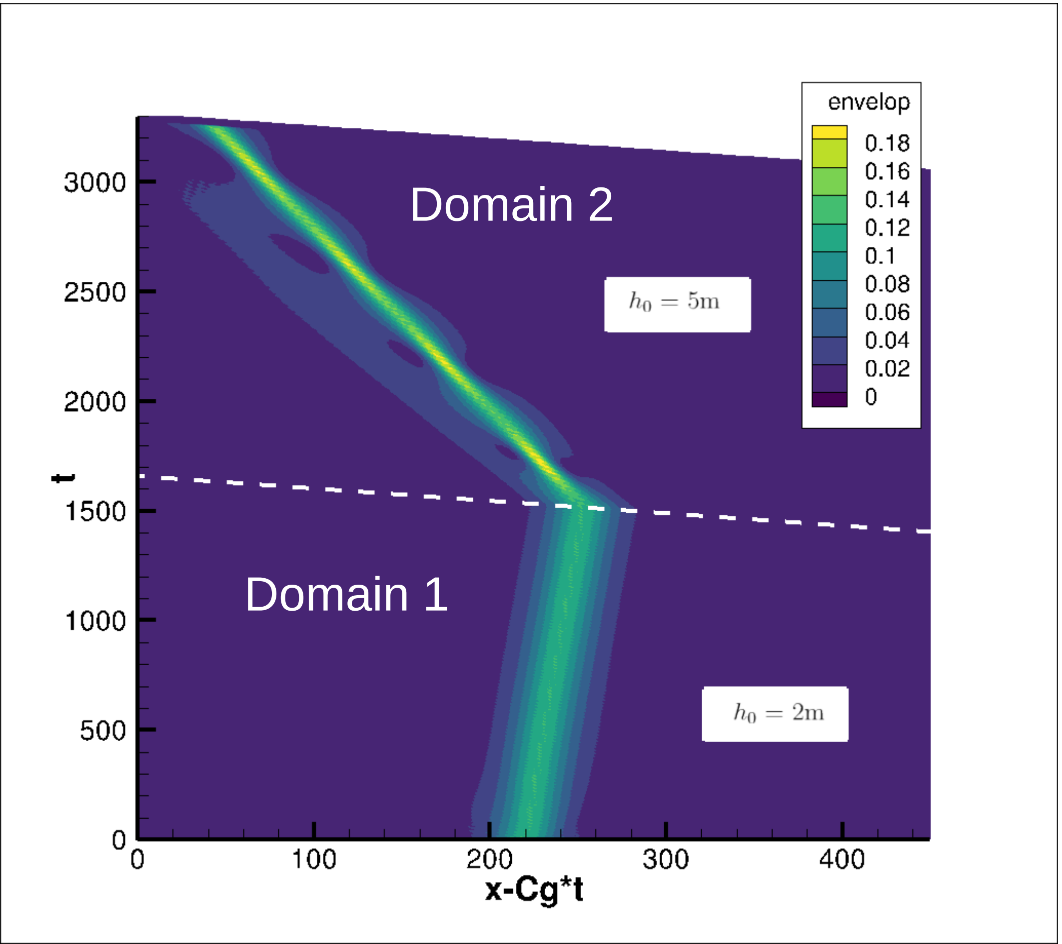
Infinitely large wave amplification is allowed



When $k_2h_2 >> 1$ and k_1h_1 tends down to the limit 1.363, the nonlinear coefficient S grows infinitely. Though the maximum amplitude of the newly generated solitons is limited by the value of $2Ta$, within the NLS theory the constructive interference of the infinite number N of solitons can course unrestrictedly large waves.

Direct num. sims. of hydrodynamical equations

We simulate the problem using the HOSM solver of the potential Euler equations [Gouin et al, 2016] taking into account strongly nonlinear effects (*M* = 8). The bottom variation should be relatively moderate and smooth, hence the step is replaced by a smooth profile.



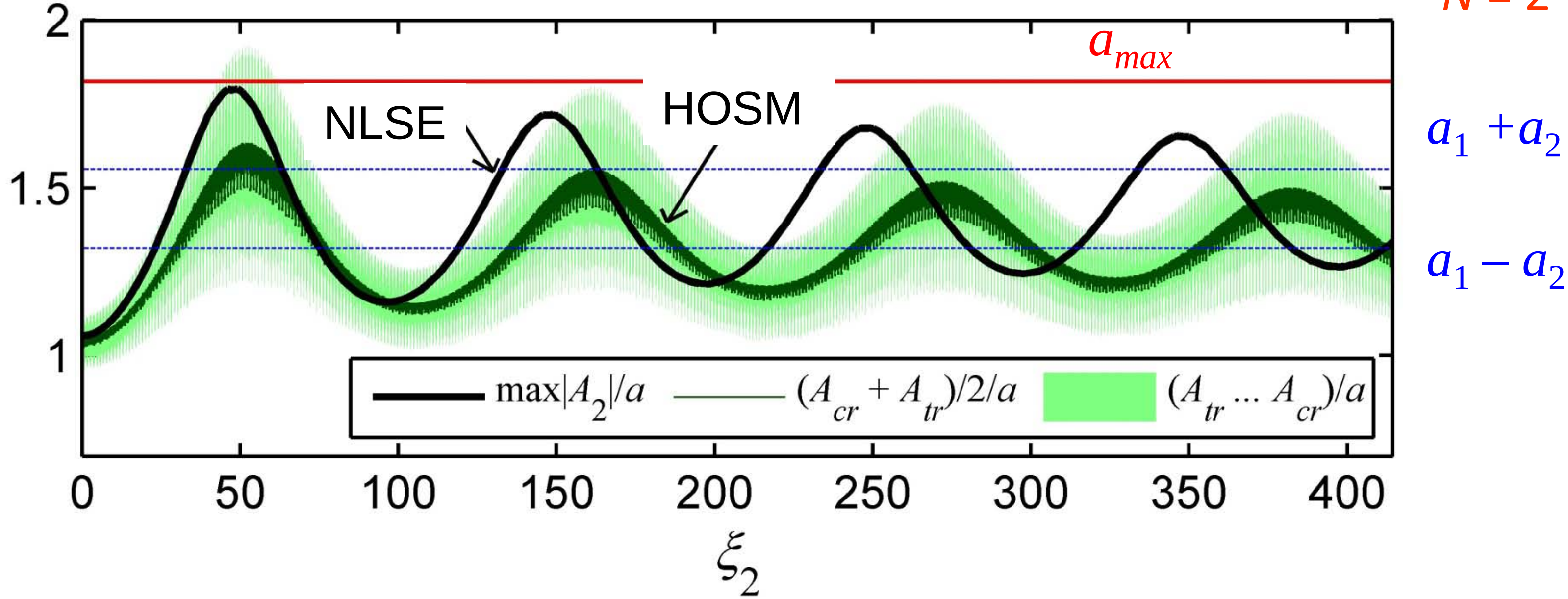
The initial condition is produced using the exact soliton solution of the NLSE. Later on the soliton group adjusts the shape when travels freely in the domain 1.

Large spatial domain is used, and an absorbing zone close to the boundaries is added to prevent from the presence of parasitic waves.

Direct num. sims. of hydrodynamical equations

Example: $k_1h_1 = 2$, $h_2/h_1 = 2.5$, $k_1a = 0.12$

$N = 2$

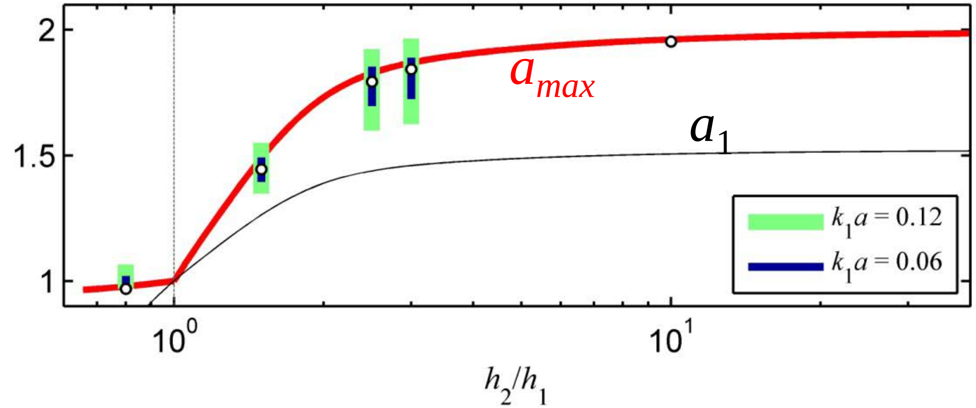


The solution of the primitive equations is reasonably well captured by the weakly nonlinear solution, including the extreme wave amplitudes.

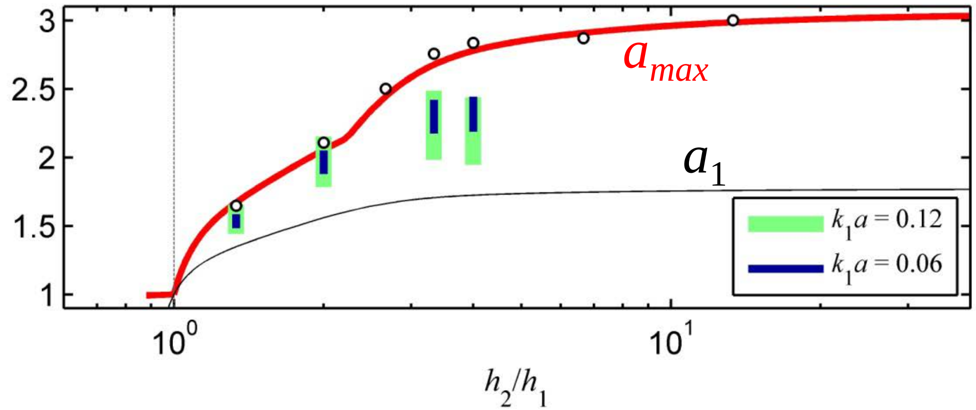
Direct num. sims. of hydrodynamical equations

Maximum wave amplitude in the second domain

$k_1 h_1 = 2$



$k_1 h_1 = 1.5$



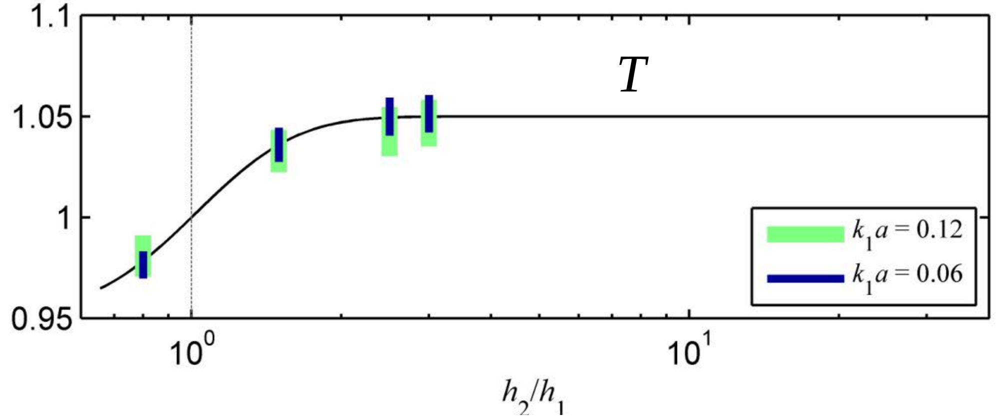
The solution of the primitive equations (bars) agree well with the direct numerical simulation of the NLSE (circles) and with the analytic approximate solution for a_{max} (the red curve).

Some disagreement in the case $k_1h_1 = 1.5$ and large h_2/h_1 may be due to the shortcomings of the numerical code (too large bottom step).

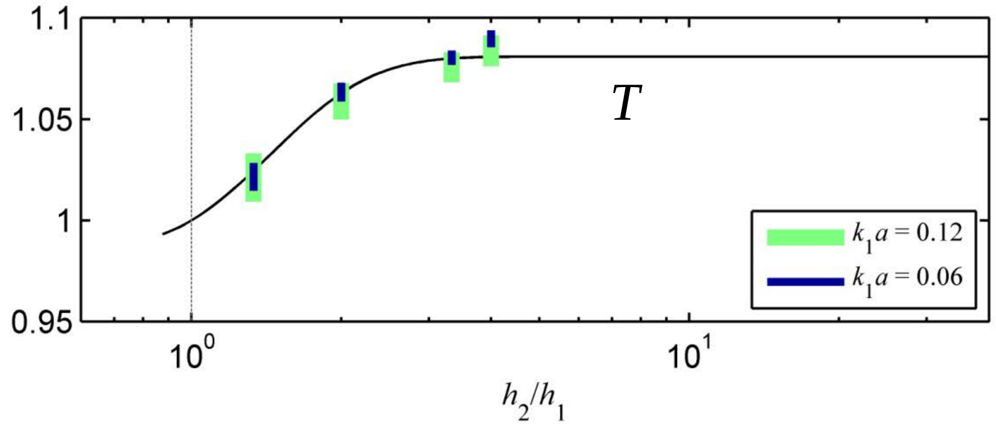
Direct num. sims. of hydrodynamical equations

Estimation of the linear transmission coefficient *T*

$k_1 h_1 = 2$



$k_1 h_1 = 1.5$



The approximate linear solution of the transmission coefficient *T* (the black curves) serves well in our case of nonlinear waves with finite spectral bandwidth (the direct numerical simulation of the Euler equations are shown with bars).

Conclusion

We show that **waves can grow** in amplitude under the condition when the **depth increases rapidly** after a relatively shallow zone

This effect is produced by specific dynamics of coherent **soliton-type nonlinear groups**

We present a weakly nonlinear solution capable to approximate the **description of water wave amplification**

We prove in DNS of the **primitive equations of hydrodynamics** that the transmitted wave amplitude **can increase more than twice** due to this effect

This effect provides a **new mechanism of rogue wave generation** in the water of intermediate depth

Similar dynamics can occur in **other systems described by the NLSE** with **rapidly changing conditions**

G. Ducrozet, A.V. Slunyaev, Y.A. Stepanyants, Transformation of envelope solitons on a bottom step. *Physics of Fluids* 33, 066606 (2021). ArXiv: 2104.11432