Complex interactions of wave breathers

Andrey Gelash

Skoltech, Moscow & Institute of Automation and Electrometry, Novosibirsk

Nonlinear Session - 2021

Nonlinear Schrödinger Equation (NLSE) and its exact solutions

$$i\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = 0$$

The NLSE is integrable using the Inverse Scattering Transform (IST). The IST is based on the auxiliary Zakharov-Shabat linear system for the **2×2** matrix wave function **Φ**:

$$\mathbf{\Phi}_x - \begin{pmatrix} -i\lambda & \psi \\ -\psi^* & i\lambda \end{pmatrix} \mathbf{\Phi} = \mathbf{0}$$

$$\mathbf{\Phi}_t - \begin{pmatrix} -i\lambda^2 + i|\psi|^2/2 & \lambda\psi + i\psi_x/2 \\ -\lambda\psi^* + i\psi_x^*/2 & i\lambda^2 - i|\psi|^2/2 \end{pmatrix} \mathbf{\Phi} = \mathbf{0}$$

λ is a complex-valued spectral parameter

Compatibility condition:

$$\mathbf{\Phi}_{xt} = \mathbf{\Phi}_{tx}$$

Solitons in a parametrically unstable plasma

E. A. Kuznetsov

Institute of Automation and Electrometry, Siberian Branch, Academy of Sciences of the USSR, Novosibirsk

(Presented by Academician R. Z. Sagdeev, June 23, 1977)
(Submitted May 2, 1977)
Dokl. Akad. Nauk SSSR **236**, 575–577 (September 1977)

PACS numbers: 52.35.Py, 52.35.Ra

$$i\psi_t + {}^1\!/_2\psi_{xx} + (|\psi|^2 - |E_0|^2)\psi = 0$$

with the boundary condition

$$\psi \to E_0 \quad \text{as} \quad |x| \to \infty.$$

$$\zeta = (\lambda^2 + |E_0|^2)^{1/2}$$

$$\lambda_{1,2} = \pm i\mu;\ \zeta_{1,2} = i\nu;$$

$$\Omega = 2\mu\nu.$$

$$E(x,t) = 2\frac{\nu}{\mu}\left(\frac{\nu\cos\Omega t + i\mu\sin\Omega t}{\operatorname{ch} 2\nu x + E_0/\mu\cos\Omega t}\right),$$

Eigenvalue problem

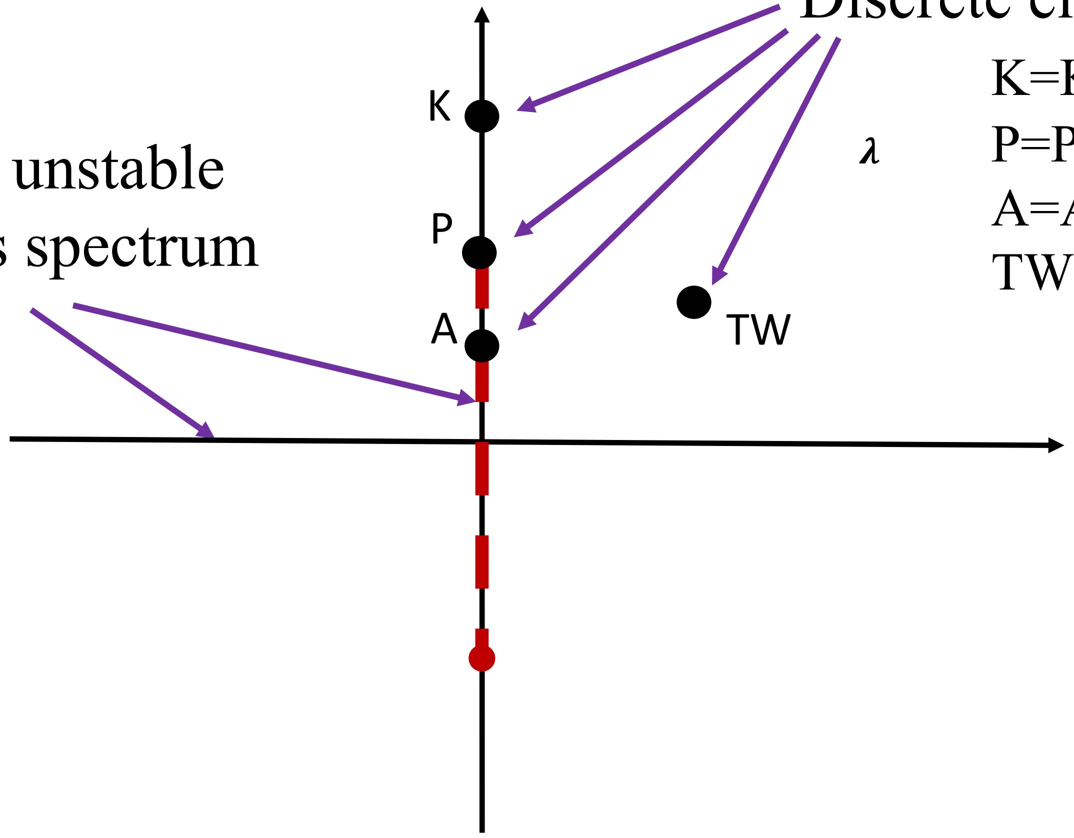
$$\hat{\boldsymbol{L}} = i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{\partial}{\partial x} - i\begin{pmatrix}0 & \psi\\ \psi^* & 0\end{pmatrix}$$

$$\hat{\boldsymbol{L}}\boldsymbol{\Phi} = \lambda\boldsymbol{\Phi},$$

Discrete eigenvalues:

K=Kuznetsov
P=Peregrine
A=Akhmediev
TW=Tajiri-Watanabe

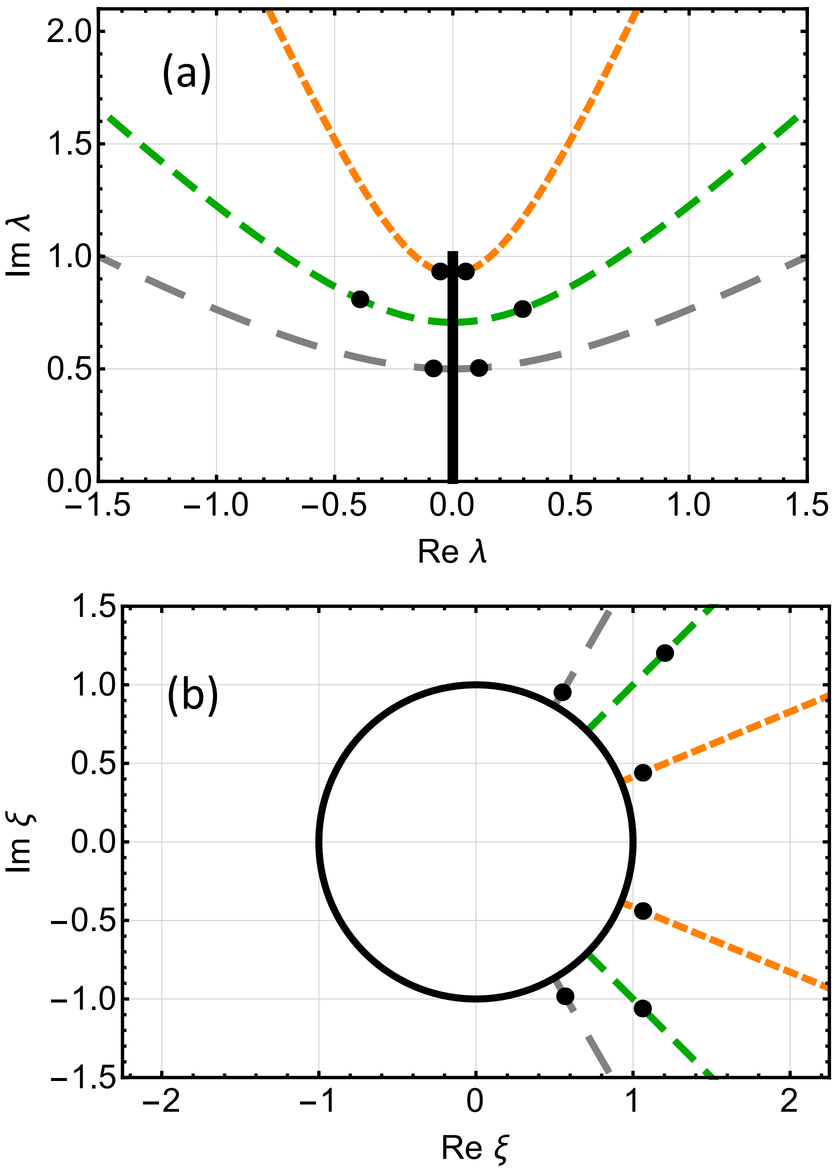
Stable and unstable continuous spectrum



$$\zeta = \sqrt{\lambda^2 + 1}$$

Branch cut $[-i, i]$

Two different parametrizations for eigenvalues



$$\lambda = \frac{i}{2}\left(\xi + \frac{1}{\xi}\right), \qquad \xi = Re^{i\alpha}$$

$$\mathrm{Re}[\lambda] = \pm\frac{\sin\alpha}{2}\left(R - \frac{1}{R}\right) \qquad \mathrm{Im}[\lambda] = \frac{\cos\alpha}{2}\left(R + \frac{1}{R}\right) \qquad (*)$$

$$\alpha_1 = -\alpha_2 \qquad (**)$$

Comparison of λ and ξ parametrizations of the spectral parameter. The branch cut and its Joukowsky mapping are drawn by black solid lines. The pairs of breather eigenvalues (marked by black points) lie on (a) the parametric curves Eq. (**) (dashed lines) and (b) the rays Eq. (*) (dashed lines).

General N-breather solution formula

$$\psi_N = e^{it}\left[1 + 2\det\begin{pmatrix} 0 & q_{1,2} & \cdots & q_{N,2} \\ q_{1,1}^* & & & \\ \vdots & & \widehat{M}^T & \\ q_{N,1}^* & & & \end{pmatrix}(\det \widehat{M})^{-1}\right]$$

$$\widehat{M}_{nm} = \frac{i(\boldsymbol{q}_n \cdot \boldsymbol{q}_m^*)}{R_n e^{i\alpha_n} + \frac{e^{-i\alpha_n}}{R_n} - R_m e^{-i\alpha_m} - \frac{e^{i\alpha_m}}{R_m}}$$

$$q_{i1} = e^{-\phi_i} - \frac{e^{\phi_i - i\alpha_i}}{R_i} \qquad q_{i2} = e^{\phi_i} - \frac{e^{-\phi_i - i\alpha_i}}{R_i}$$

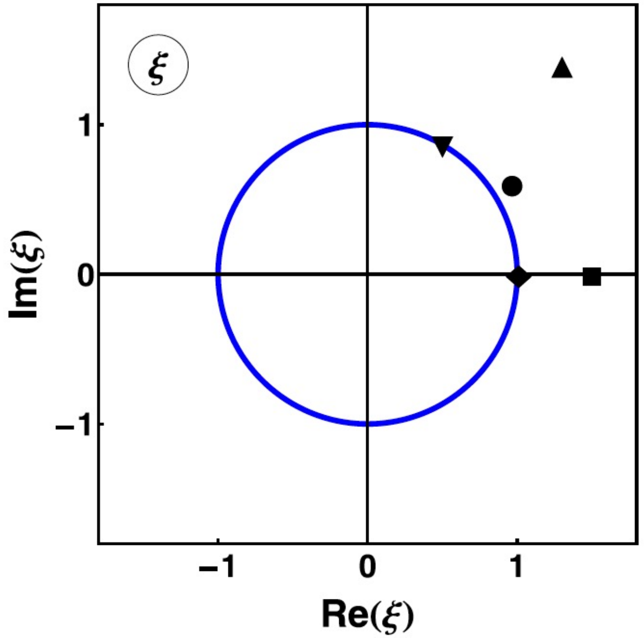
$$\phi_i = \eta_i(x - x_{0,i}) + \gamma_i t + i\left(k_i x + \delta_i t - \frac{\theta_i}{2}\right)$$

$$\eta_i = -\frac{1}{2}\left(R_i - \frac{1}{R_i}\right)\cos\alpha_i \qquad \gamma_i = -\frac{1}{4}\left(R_i^2 + \frac{1}{R_i^2}\right)\sin 2\alpha_i$$

$$k_i = -\frac{1}{2}\left(R_i + \frac{1}{R_i}\right)\sin\alpha_i \qquad \delta_i = \frac{1}{4}\left(R_i^2 - \frac{1}{R_i^2}\right)\cos 2\alpha_i$$

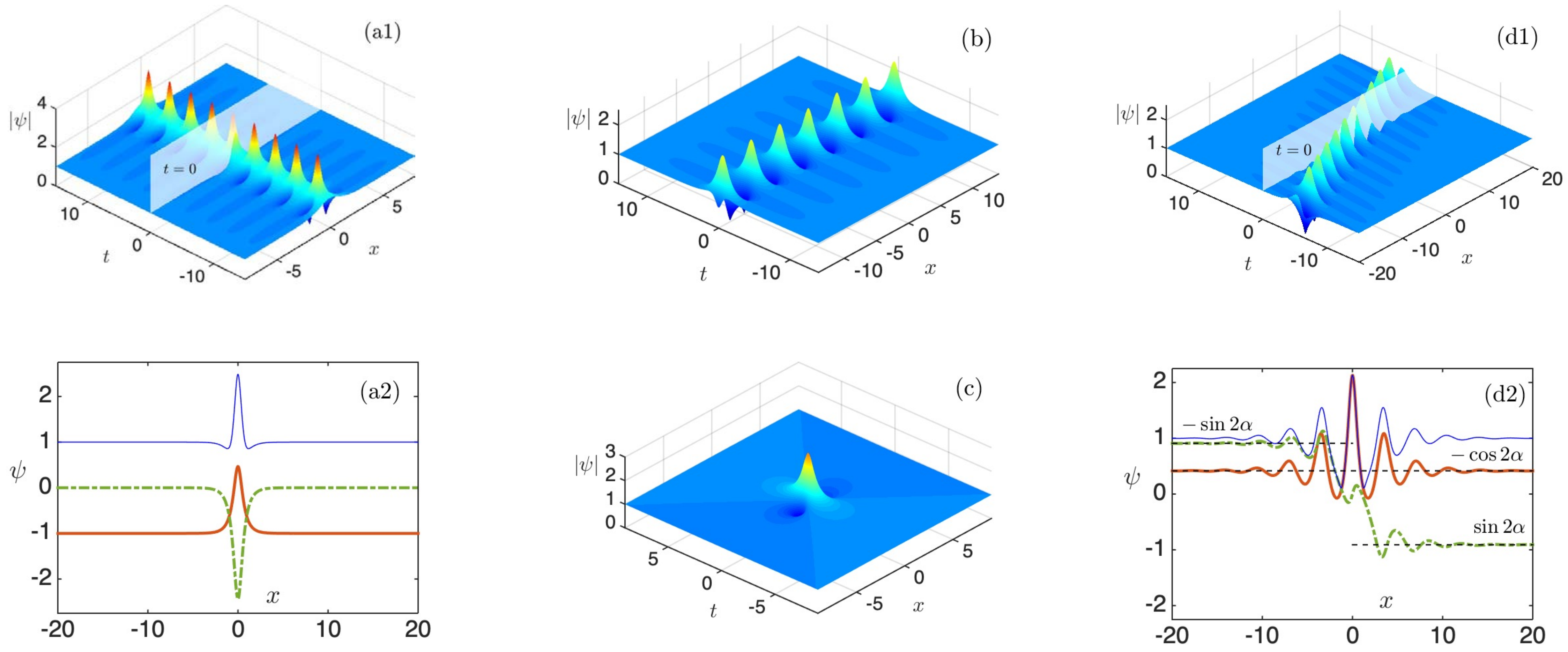
One breather has four real-valued parameters:
R_i, α_i - breather amplitude and group velocity
$x_{0,i}$, θ_i - position and phase

$$\xi = Re^{i\alpha}$$



▲ - Tajiri-Watanabe,
▼ - Akhmediev breather, ● - quasi-Akhmediev breather
■ - Kuznetsov breather, ◆ - Peregrine breather

Fundamental one-breather solutions

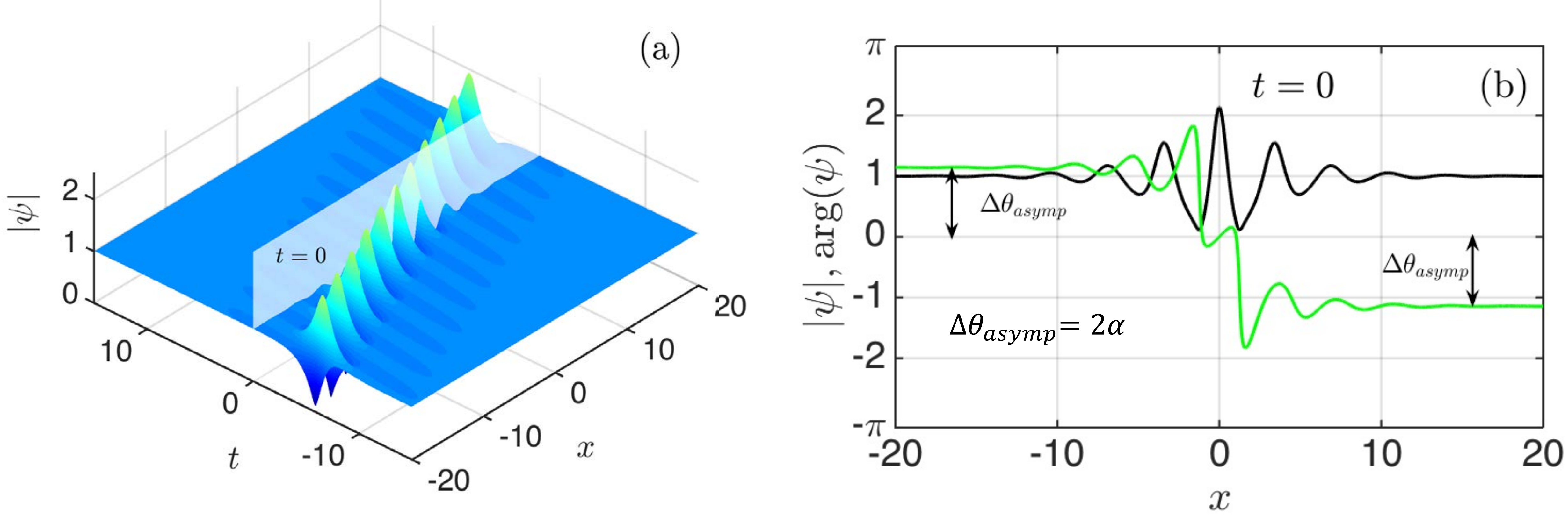


(a) Kuznetsov breather

(b) Akhmediev breather
(c) Peregrine breather

(d) Tajiri-Watanabe breather

General single-breather solution: $R > 1$, $\alpha > 0$

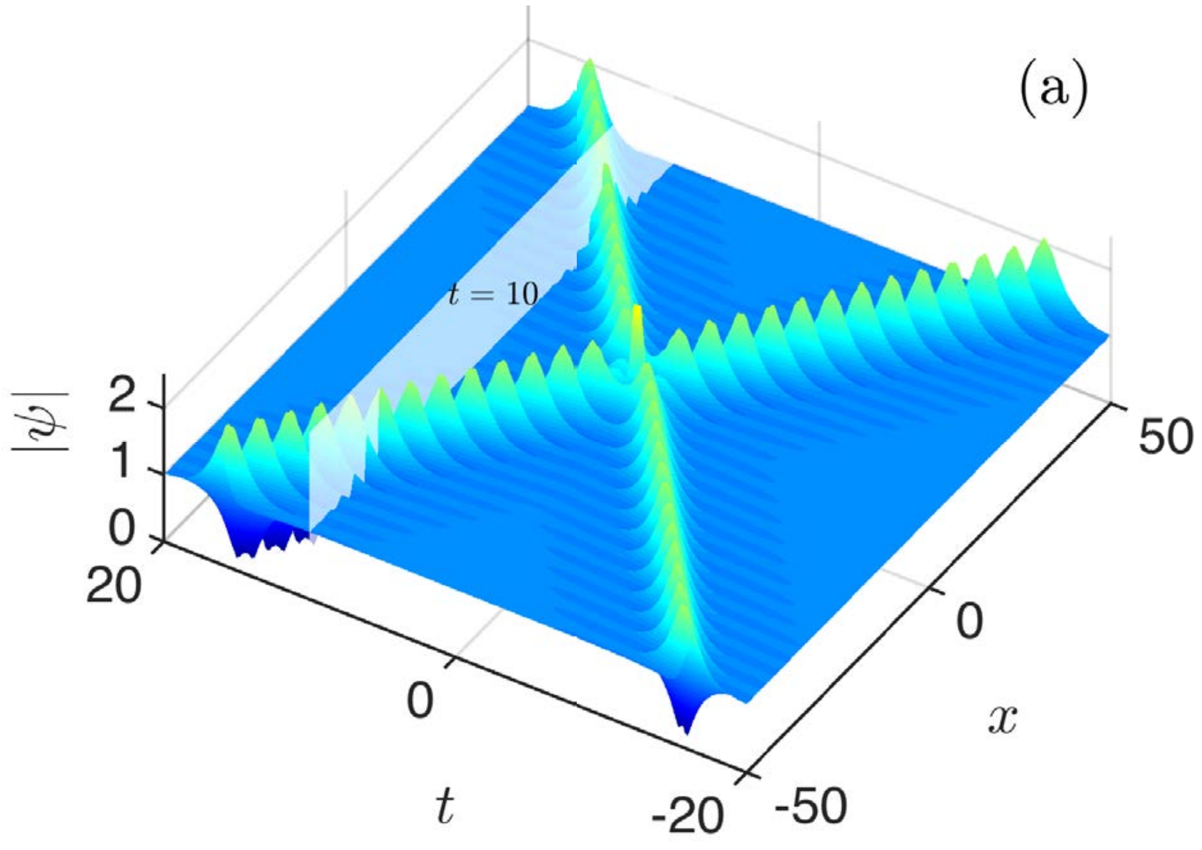


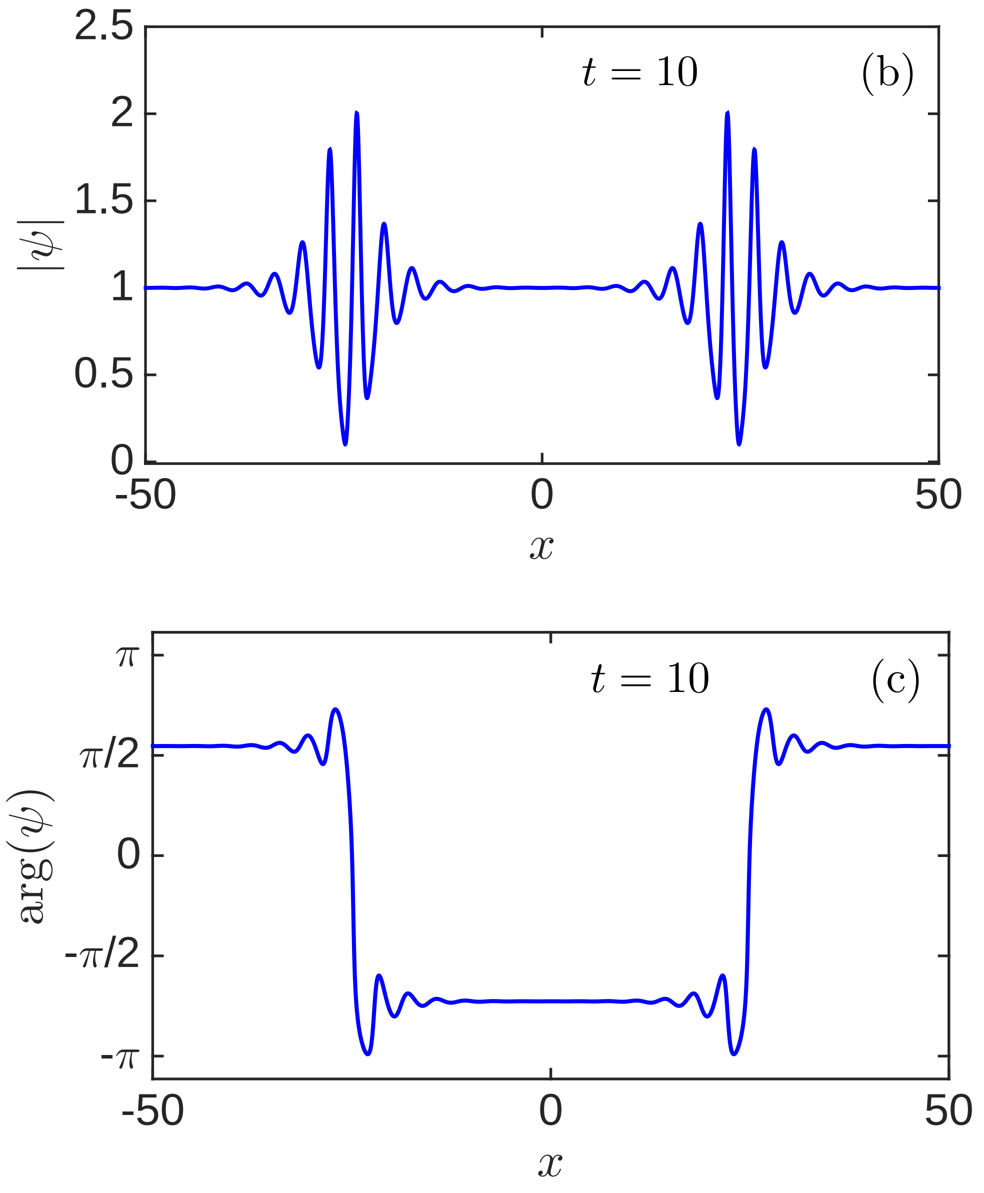
$$\psi = \left(1 + 2\left(R + {}^{1}/_{R}\right)\cos\alpha \frac{q_1^* q_2}{|q_1|^2 + |q_2|^2}\right) e^{it} e^{i\theta_c}$$

$$V_{gr} = -\sin\alpha\,(R^4 + 1)/\big(R(R^2 - 1)\big)$$

$$V_{ph} = -\frac{1}{2}\left(R^2 - \frac{1}{R^2}\right)\cos 2\alpha$$

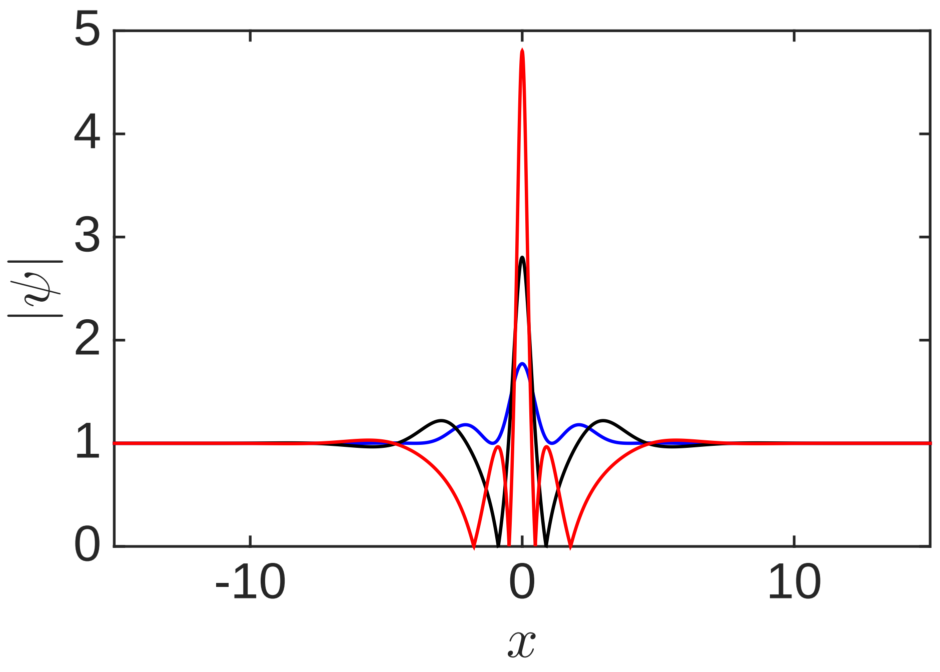
Two-breather solution with $R_1 = R_2 = R$; $\alpha_1 = \alpha$; $\alpha_2 = -\alpha$; $R > 1, \alpha > 0$.





Typical two-breather collision and its asymptotic state. (a) shows spatial-temporal portrait of a typical two-breather solution with $R = 1.35$, $\alpha = 1.0$, $\theta_1 = 0$, $\theta_1 = 0.025$. Blue solid lines in (b) and (c) shows spatial profile of the solution at $\xi = 10$.

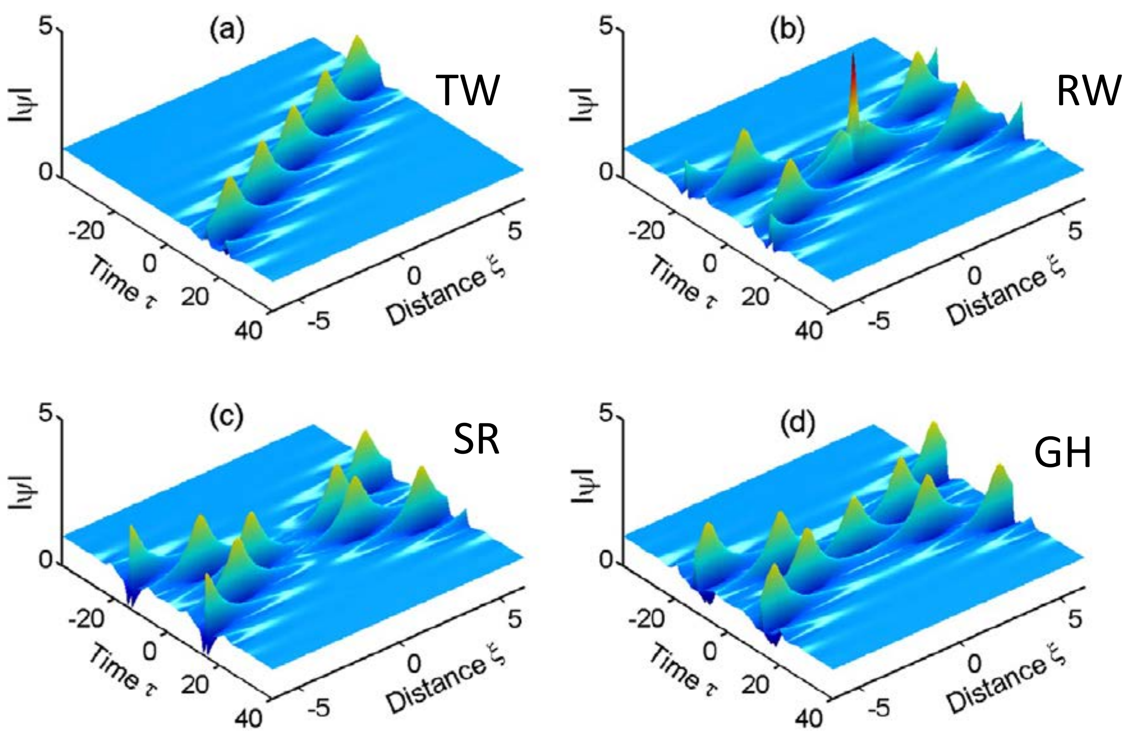
Phase synchronization of two-breather collision



Amplitude profiles at t=0 for the basic cases of two-breather collision: superregular (SR, blue line), rogue wave (RW, red line) and ghost (GH, black line) phase synchronizations.

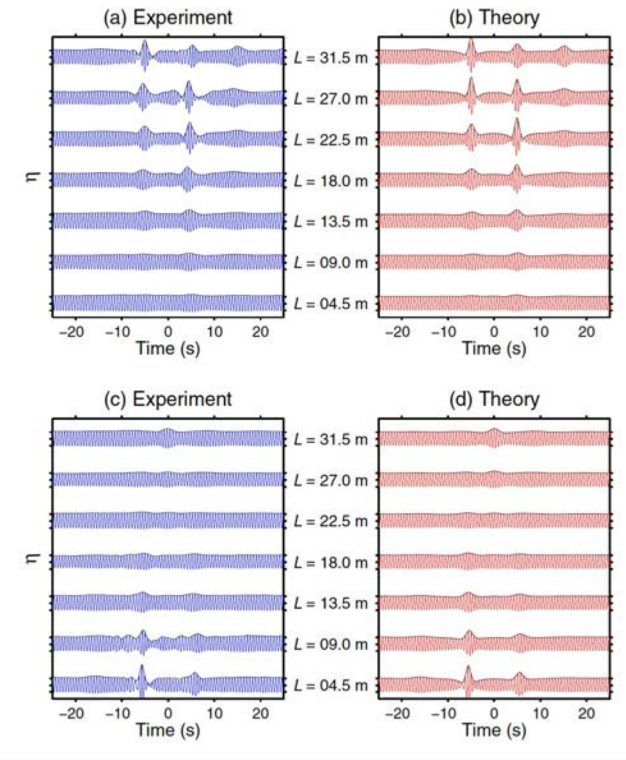
$$\theta_{\rm SR} = \{\pi/2; 3\pi/2\}, \quad \theta_{\rm RW} = 0, \quad \theta_{\rm GH} = \pi.$$

[1,2]



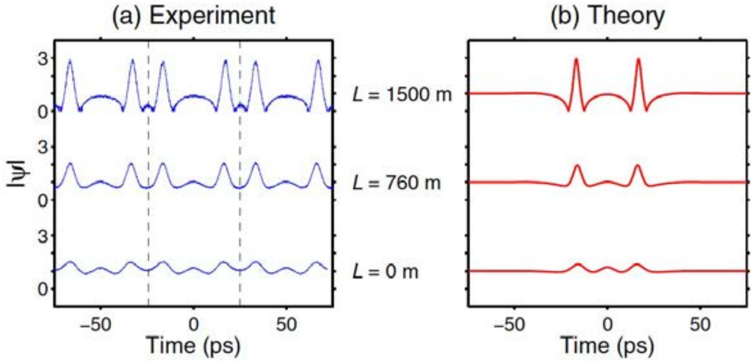
Experimental observation of SR and GH breather interactions

Water waves



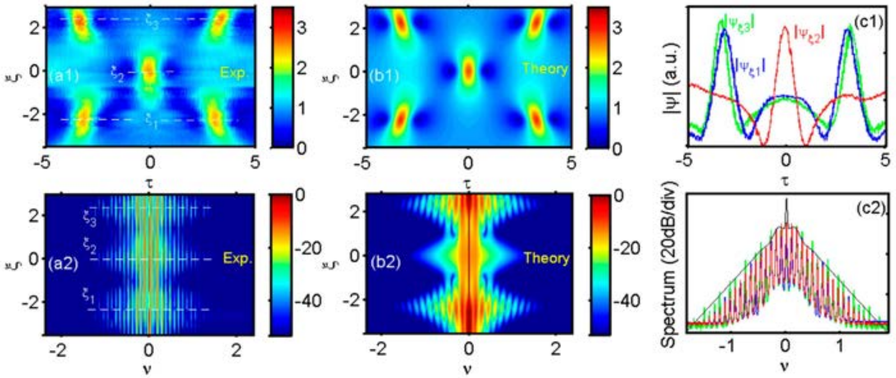
[2]

Light waves

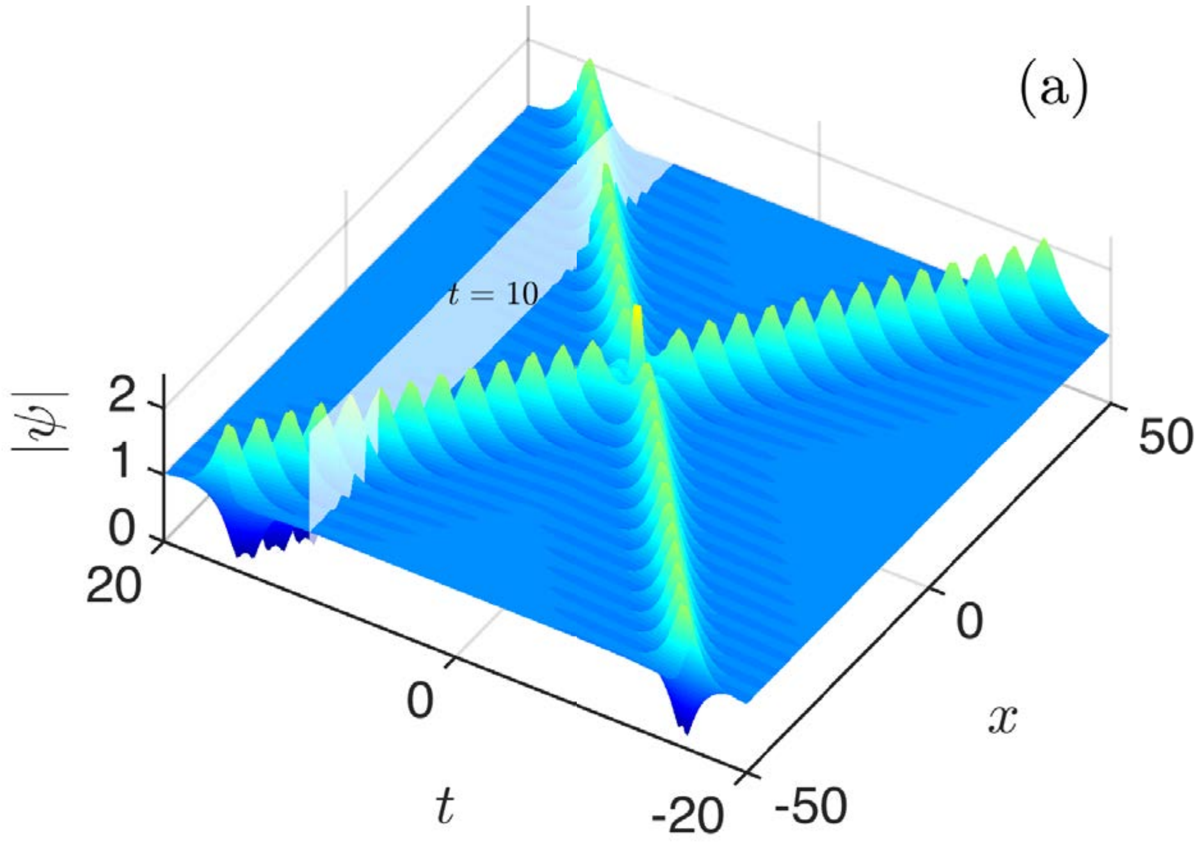


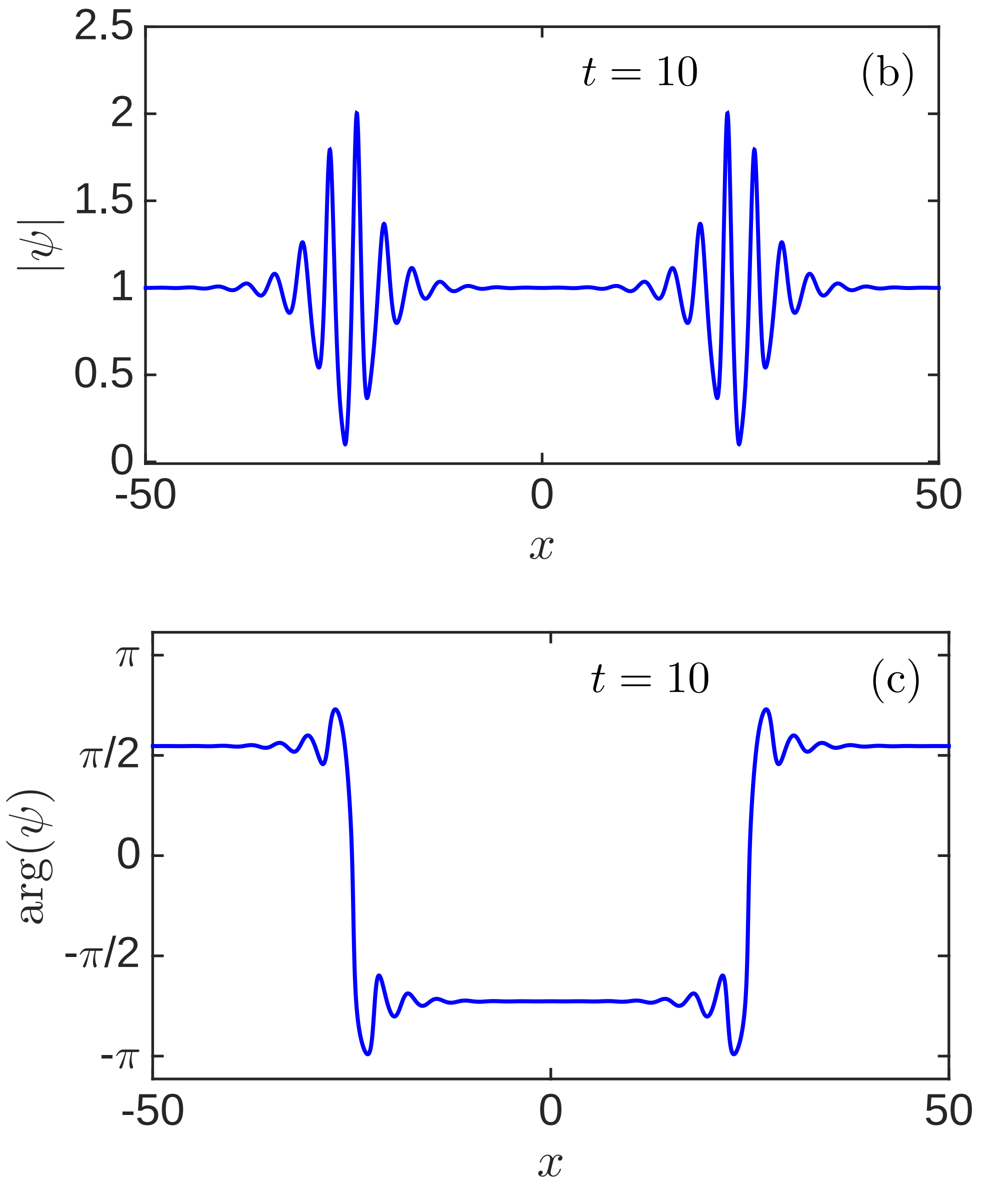
[3]

Light waves



Two-breather solution with $R_1 = R_2 = R$; $\alpha_1 = \alpha$; $\alpha_2 = -\alpha$; $R > 1, \alpha > 0$.





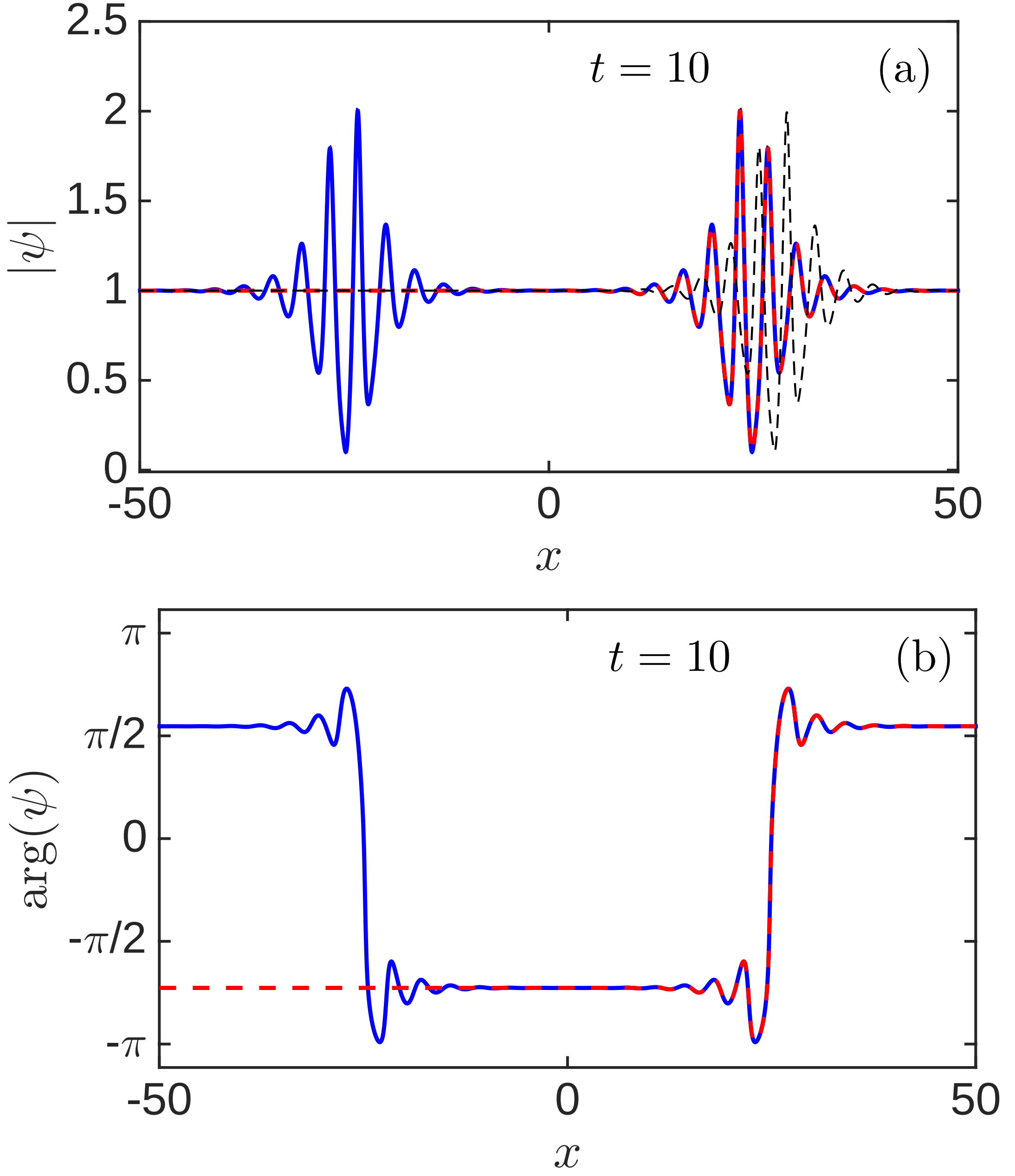
Typical two-breather collision and its asymptotic state. (a) shows spatial-temporal portrait of a typical two-breather solution with $R = 1.35$, $\alpha = 1.0$, $\theta_1 = 0$, $\theta_1 = 0.025$. Blue solid lines in (b) and (c) shows spatial profile of the solution at $\xi = 10$.

Space-phase shifts formulas [4]

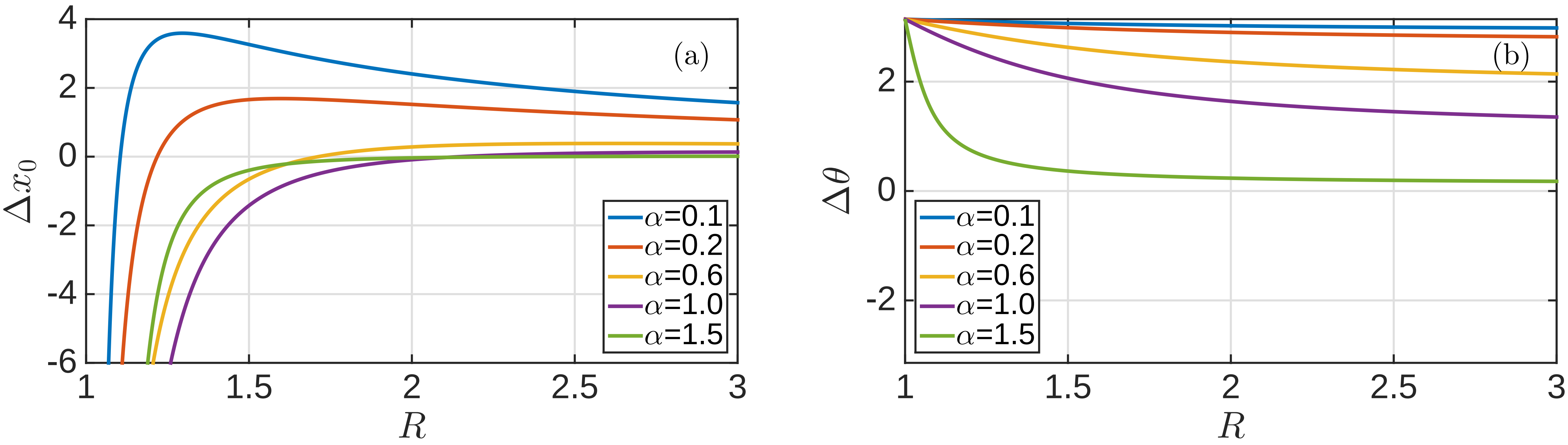
$$\Delta x_0 = \frac{\ln\left[\dfrac{\left(R-\frac{1}{R}\right)^2}{\sin^2\alpha\left(R^2+\frac{1}{R^2}+2\cos 2\alpha\right)}\right]}{\left(R-\frac{1}{R}\right)\cos\alpha}$$

$$\Delta\theta_0 = -2\arg\left[\frac{2\,R^3(1-R^2)\tan\alpha}{(1+R^2)^2\cos\alpha+i(1-R^4)^2\sin\alpha}\right]$$

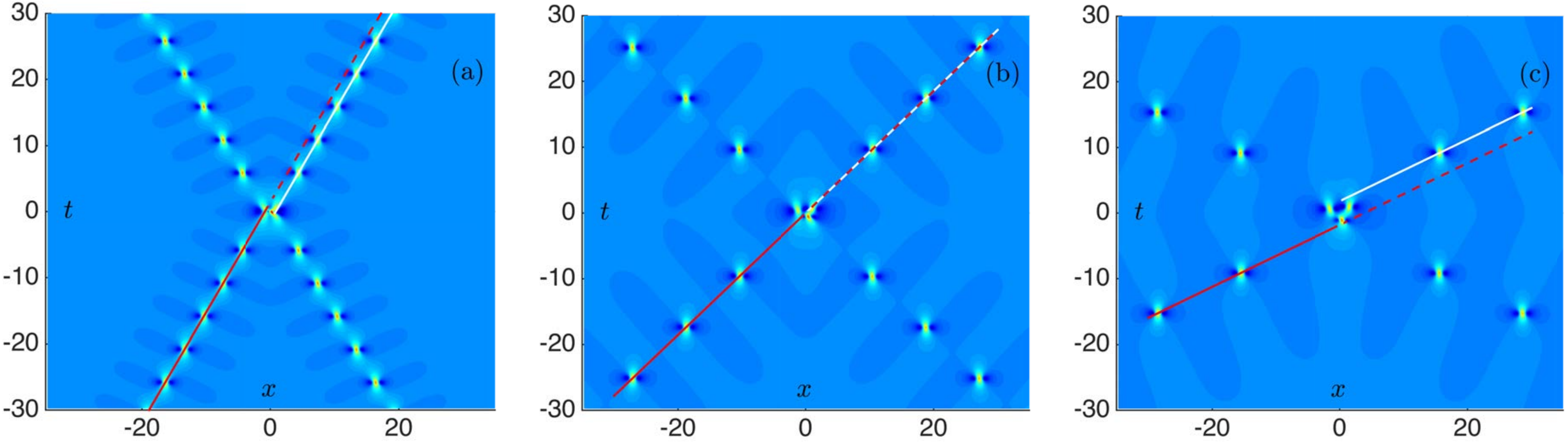
$$\Delta\theta_c = 4\,\alpha$$



Dependence of space and phase shifts on breather parameters



Positive, zero and negative space shifts acquiring by breathers after collision.



Space-phase shifts formulas in general case [5]

$$\Delta x_{0,21} = ln\big[(s_1 - s_3)/(s_2 - s_4)\big]/2Im[\zeta_2]\,,$$
$$\Delta\theta_{21} = 2\mathrm{Arg}[i(p_1 + p_2)]\,, \Delta\theta_{c,21} = -4\mathrm{Arg}[\lambda_1 + \zeta_1]\,.$$

$$\begin{aligned}
s_1 &= (A^4 + |\lambda_1 + \zeta_1|^2 \cdot |\lambda_2 + \zeta_2|^2) \cdot |\lambda_1 - \lambda_2^*|^2 + \\
&\quad + A^2(|\lambda_1 + \zeta_1|^2 + |\lambda_2 + \zeta_2|^2) \cdot |\lambda_1 - \lambda_2|^2\,, \\
s_2 &= A^2(|\lambda_1 + \zeta_1|^2 + |\lambda_2 + \zeta_2|^2) \cdot |\lambda_1 - \lambda_2^*|^2 + \\
&\quad + (A^4 + |\lambda_1 + \zeta_1|^2 \cdot |\lambda_2 + \zeta_2|^2) \cdot |\lambda_1 - \lambda_2|^2\,, \\
s_3 &= A^2(\lambda_1 - \lambda_1^*)(\lambda_2 - \lambda_2^*) \cdot [(\lambda_1 + \zeta_1)(\lambda_2 + \zeta_2) + \\
&\quad + (\lambda_1^* + \zeta_1^*)(\lambda_2^* + \zeta_2^*)]\,, \\
s_4 &= -A^2(\lambda_1 - \lambda_1^*)(\lambda_2 - \lambda_2^*) \cdot [(\lambda_1 + \zeta_1)(\lambda_2^* + \zeta_2^*) + \\
&\quad + (\lambda_1^* + \zeta_1^*)(\lambda_2 + \zeta_2)]\,; \\
p_1 &= \{A^2(\lambda_2 + \zeta_2 - \lambda_1^* - \zeta_1^*) - |\lambda_1 + \zeta_1|^2(\lambda_2^* + \zeta_2^*) + \\
&\quad + |\lambda_2 + \zeta_2|^2(\lambda_1 + \zeta_1)\}/\{|\lambda_1 - \lambda_2^*|^2\}\,, \\
p_2 &= \frac{(A^2 + |\lambda_1 + \zeta_1|^2)(\lambda_2 + \zeta_2 - \lambda_2^* - \zeta_2^*)}{(\lambda_1 - \lambda_1^*)(\lambda_2 - \lambda_2^*)}\,.
\end{aligned}$$

Well-known limit of solitons on zero background:

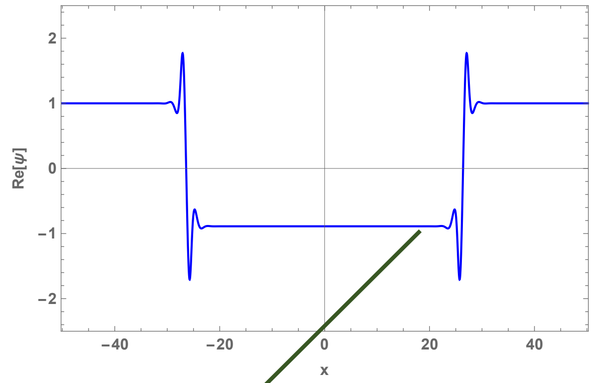
$$\lim_{A\to 0} \Delta x_{0,21} = \frac{1}{2\,Im[\lambda_2]} \ln\left(\frac{|\lambda_1 - \lambda_2^*|^2}{|\lambda_1 - \lambda_2|^2}\right)$$

$$\lim_{A\to 0} \Delta\theta_{0,21} + \Delta\theta_{c,21} + = -2\ \mathrm{Arg}\left[\frac{\lambda_1 - \lambda_2}{\lambda_1^* - \lambda_2}\right]$$

Some details of the calculations

$$\psi_2 = \mathrm{A} + 2\,\frac{\det\begin{pmatrix} 0 & q_{1,2} & q_{2,2} \\ q_{1,1}^* & i\,\dfrac{|\boldsymbol{q}_1|^2}{\lambda_1-\lambda_1^*} & i\,\dfrac{(\boldsymbol{q}_2\cdot\boldsymbol{q}_1^*)}{\lambda_2-\lambda_1^*} \\ q_{1,N}^* & i\,\dfrac{(\boldsymbol{q}_1\cdot\boldsymbol{q}_2^*)}{\lambda_1-\lambda_2^*} & i\,\dfrac{|\boldsymbol{q}_2|^2}{\lambda_2-\lambda_2^*} \end{pmatrix}}{\det\begin{pmatrix} i\,\dfrac{|\boldsymbol{q}_1|^2}{\lambda_1-\lambda_1^*} & i\,\dfrac{(\boldsymbol{q}_1\cdot\boldsymbol{q}_2^*)}{\lambda_1-\lambda_2^*} \\ i\,\dfrac{(\boldsymbol{q}_2\cdot\boldsymbol{q}_1^*)}{\lambda_2-\lambda_1^*} & i\,\dfrac{|\boldsymbol{q}_2|^2}{\lambda_2-\lambda_2^*} \end{pmatrix}}$$

General two-breather solution



$$\psi_2 = e^{-i\theta_{2c}}\left(A + 2i(\lambda_2+\lambda_2^*)\,\frac{\tilde{q}_{2,1}^*\tilde{q}_{2,2}}{|\tilde{\boldsymbol{q}}_2|^2}\right)$$

Asymptotic one-breather solution

$$q_{n,1} = e^{-\phi_n} - \frac{iAe^{\phi_n}}{\lambda_n+\zeta_n}, \qquad q_{n,2} = -\frac{iAe^{-\phi_n}}{\lambda_n+\zeta_n} + e^{\phi_n},$$

$$\phi_n = -i\zeta_n x - Im[\zeta_n]x_{0,n} - i\lambda_n\zeta_n t - i\theta_n/2$$

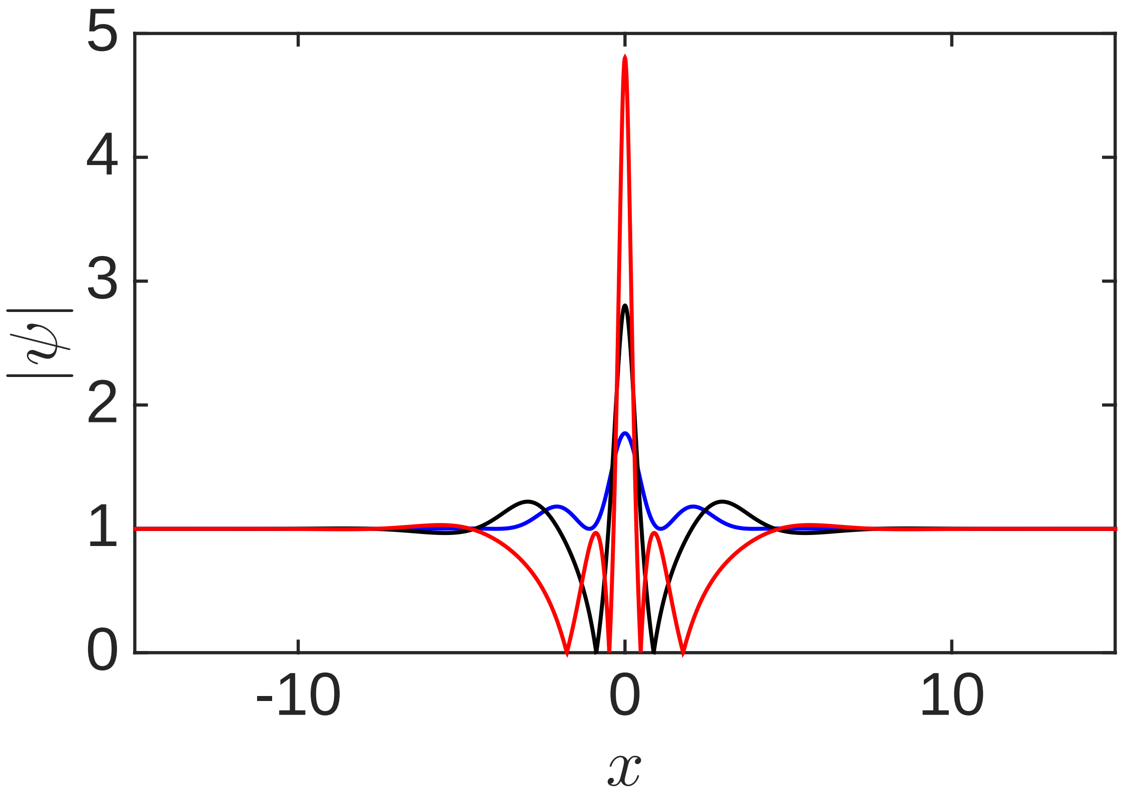
$$\zeta_n = \sqrt{\lambda_n^2 + A^2}$$

$t \to \infty, \qquad \mathrm{x} \sim t\,V_{gr2}$

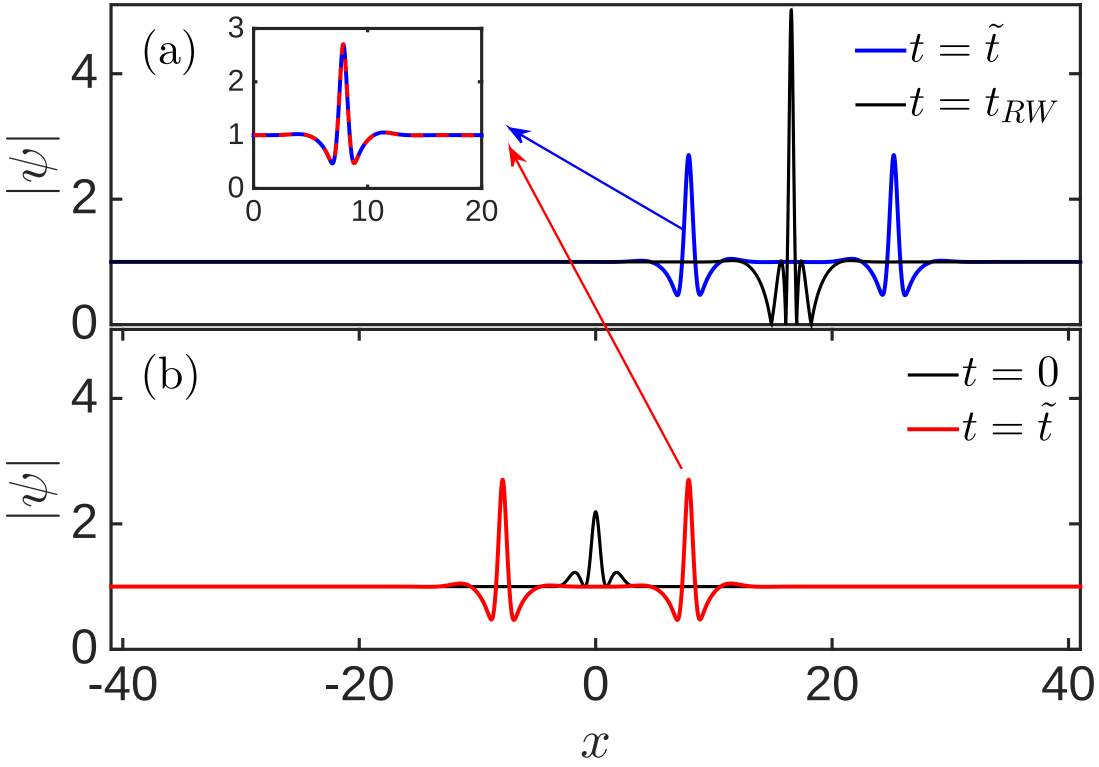
$e^{\phi_1} \to \infty, \quad e^{-\phi_1} \to 0$

$$\tilde{\boldsymbol{q}}_n(x,t,x_{0,n},\theta_n) = \\ = \boldsymbol{q}_n(x,t,x_{0,n}+\Delta x_{0,n},\theta_n+\Delta\theta_n)$$

Main idea of synchronization of breather interactions



Amplitude profiles of symmetric two-breather collisions at t = 0. Blue – SR synchronization, RW – rogue wave synchronization.



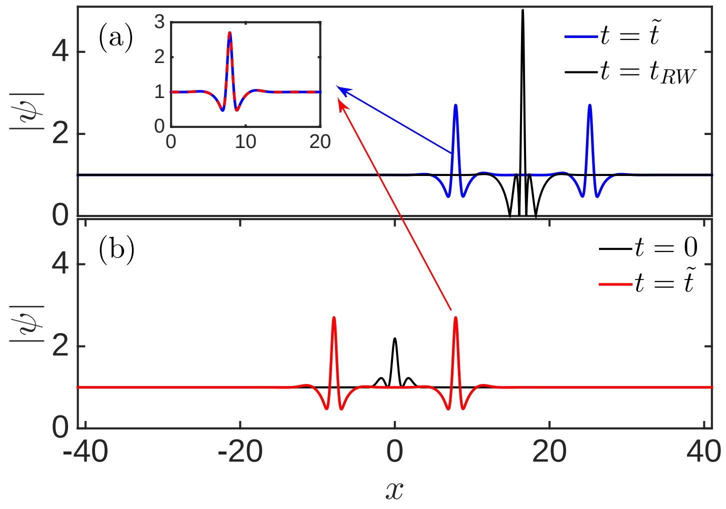
$$\phi = \eta(x - x_0) + \gamma t + i\left(kx + \delta t - \frac{\theta}{2}\right)$$

$$V_{gr} = -\frac{\gamma}{\eta} \qquad V_{ph} = -2\delta$$

Rogue wave at δx_{synch} and $t_{RW} = t_0$:

$$\theta = -V_{ph} t_0 + 2k\delta x_{synch}$$

$$x_0 = -V_{gr} t_0 + \delta x_{synch}$$



$$\phi_{2Br}^{RW} = \eta(x + V_{gr}\, t_0 - \delta x_{synch}) + \gamma t + i\left(kx + \delta t + \frac{V_{gr} t_0 - 2k\delta x_{synch} + \theta_i}{2}\right)$$

$$\phi_{1Br}^{RW} = \phi_{2Br}^{RW}(x_0 \to x_0 - \Delta x_0, \theta \to \theta - \Delta\theta) = \eta(x + \Delta x_0 + V_{gr}\, t_0 - \delta x_{synch}) + \gamma t + i\left(kx + \delta t + \frac{V_{gr} t_0 + \Delta\theta - 2k\delta x_{synch}}{2}\right)$$

$$\phi_{2Br}^{SR} = \eta x + \gamma t + i\left(kx + \delta t - \frac{\pi/2}{2}\right)$$

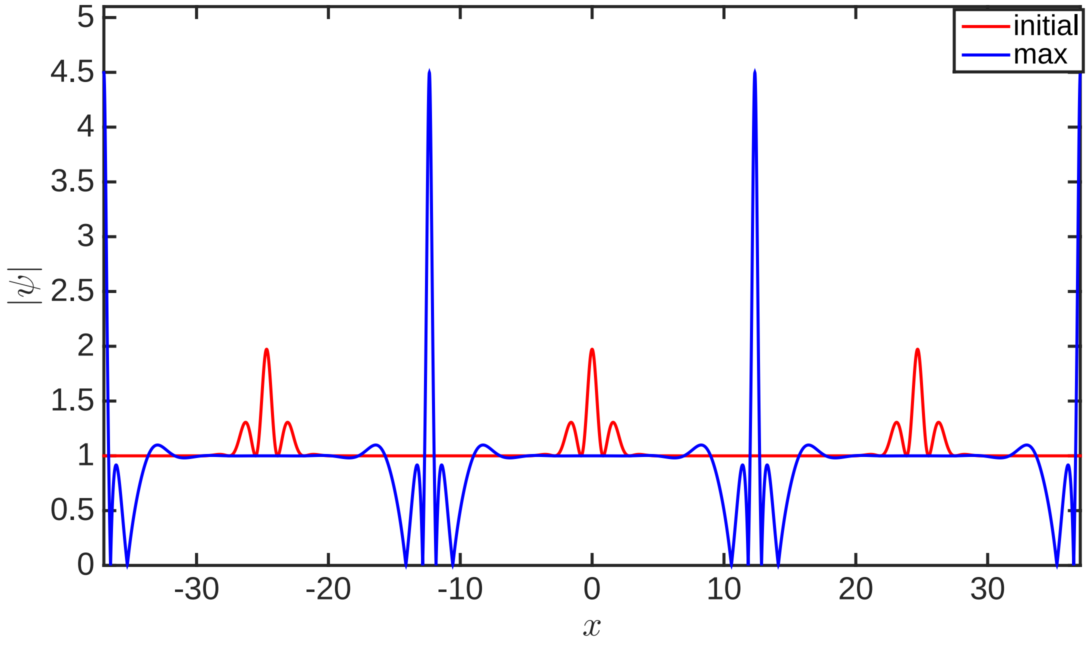
$$\phi_{1Br}^{SR} = \phi_{2Br}^{SR}(x_0 \to x_0 + \Delta x_0, \theta \to \theta + \Delta\theta) = \eta(x - \Delta x_0) + \gamma t + i\left(kx + \delta t - \frac{\theta_f}{2}\right)$$

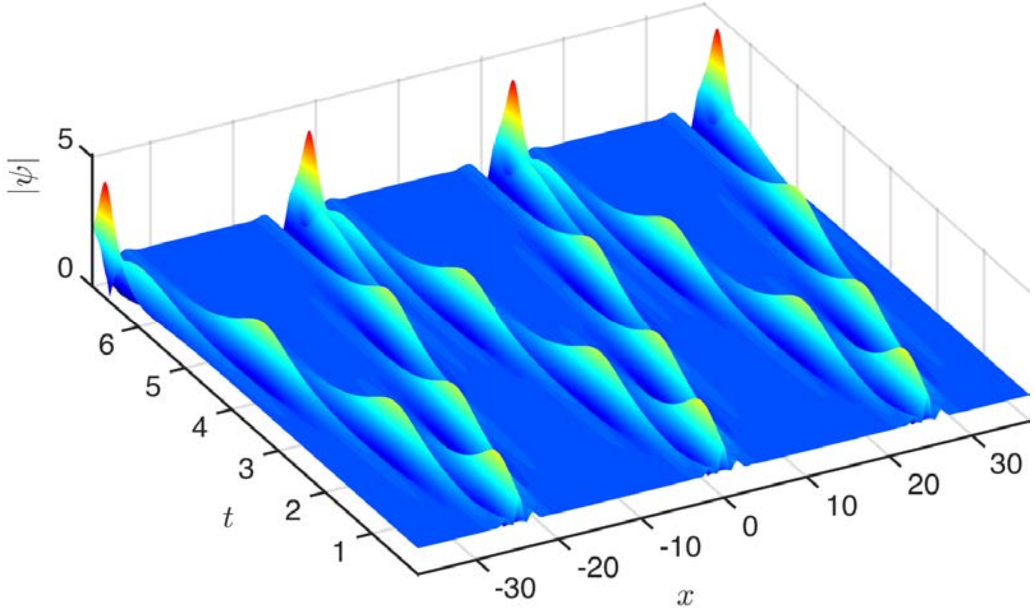
$$\phi_{1Br}^{RW} = \phi_{1Br}^{SR}$$

Synchronization

$$\delta x_{synch} = \frac{2\Delta x_0 + (2\Delta\theta_0 + \theta_i + \theta_f + 2\pi n)\frac{V_{gr}}{V_{ph}}}{1 - 2k\,\frac{V_{gr}}{V_{ph}}}$$

Synchronization: periodic train of SR perturbations

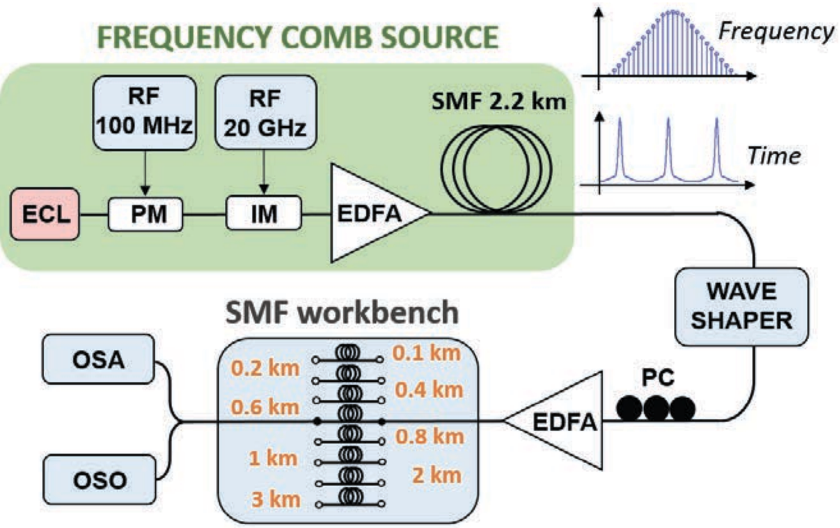


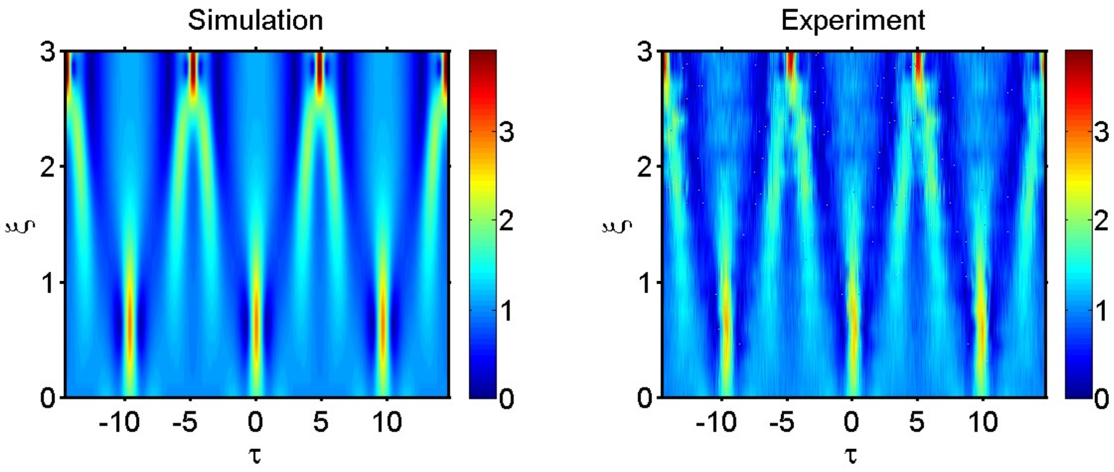


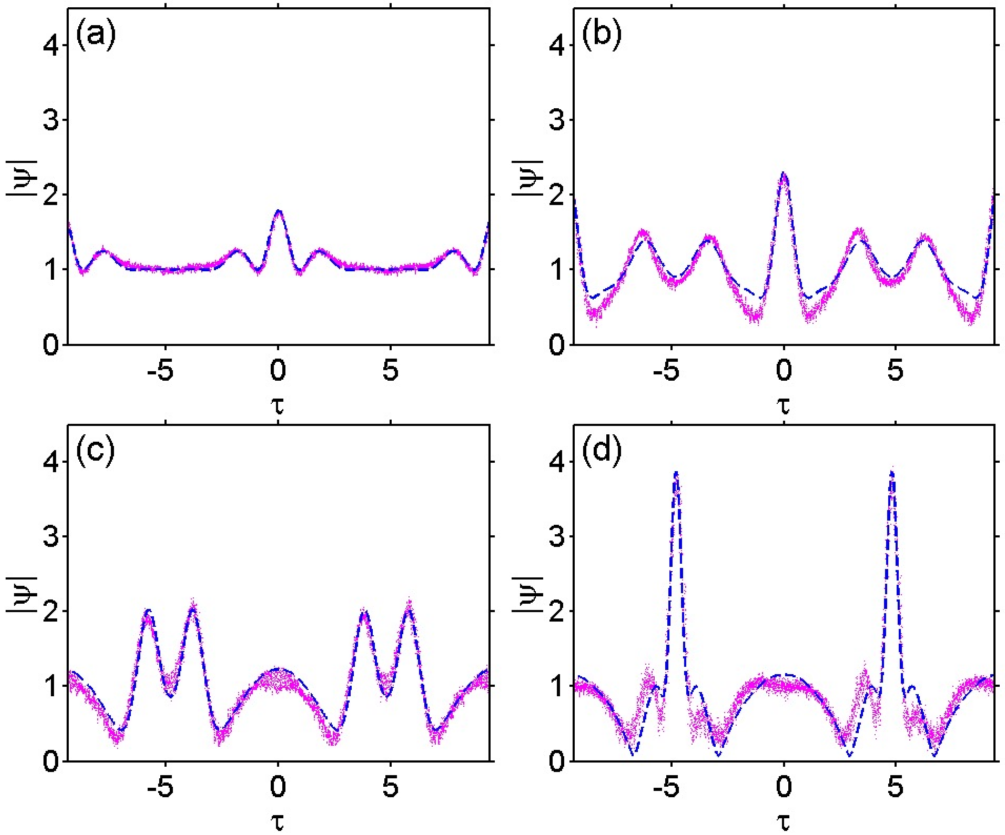
$$V_{gr} = V_{gr\_2} = \sin\alpha\,(R^4+1)/(R(R^2-1))$$
$$V_{ph} = V_{ph\_2} = -\frac{1}{2}\left(R^2 - \frac{1}{R^2}\right)\cos 2\alpha$$
$$k = -\frac{1}{2}\left(R + \frac{1}{R}\right)\sin\alpha,$$

$$\delta x_{synch} = \frac{2\Delta x_0 + \left(2\Delta\theta_0 + \theta_i + \theta_f + 2\pi n\right)\frac{V_{gr}}{V_{ph}}}{1 - 2k\,\frac{V_{gr}}{V_{ph}}}$$

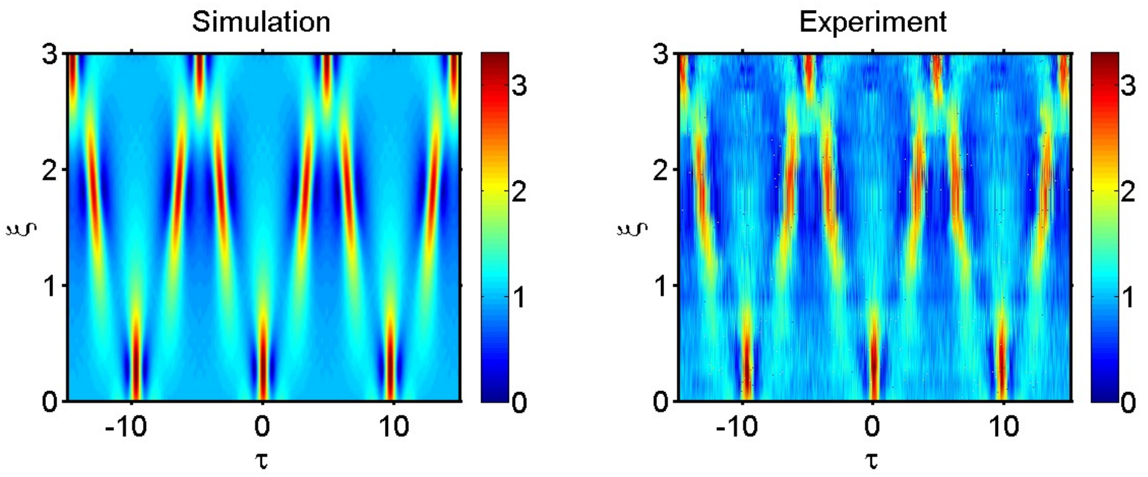
Synchronization in optical fibre experiments (Dr. Bertrand Kibler, Dr. Gang Xu, university of Dijon, France) [5]

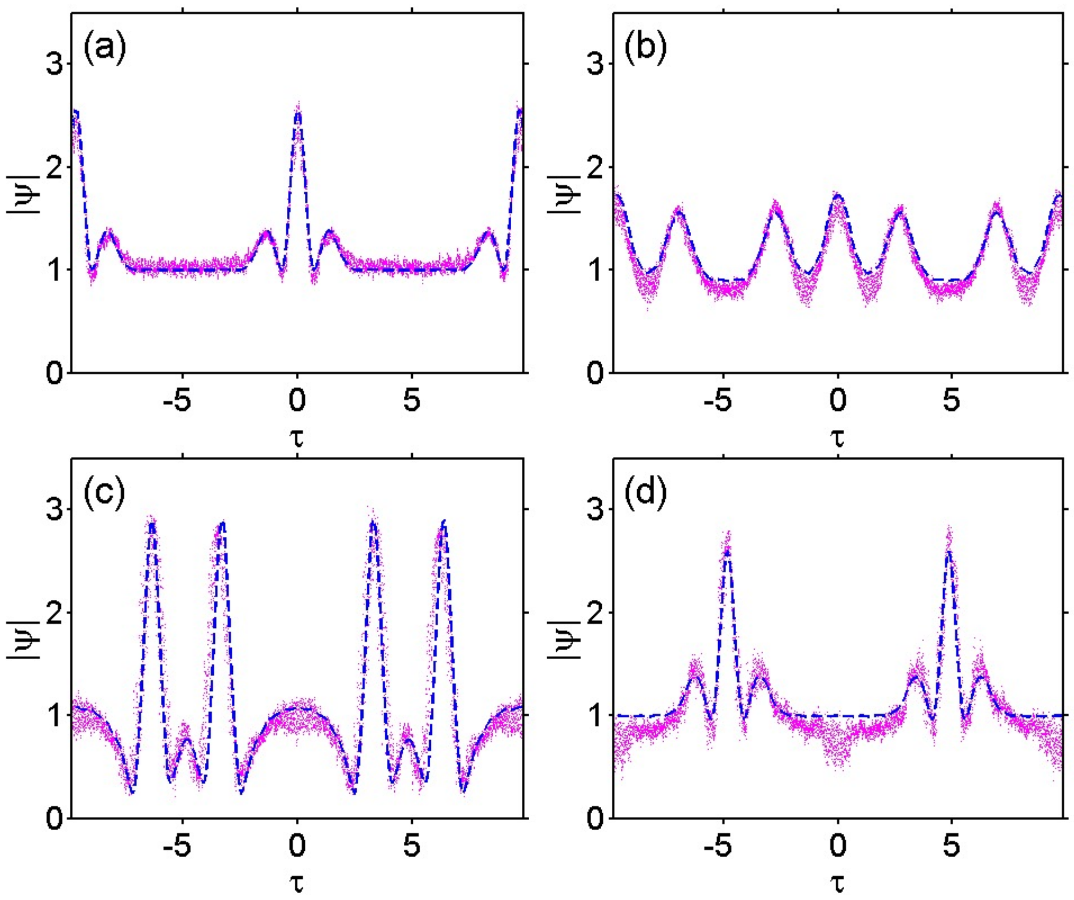






Synchronization IIIb: recurrence





References:

- [1] V. E. Zakharov and A.A. Gelash, Nonlinear stage of modulation instability. Phys. Rev. Lett. 111,054101, 2013.
- [2] B. Kibler, A. Chabchoub, A. Gelash, N. Akhmediev, and V. E. Zakharov. Superregular breathers in optics and hydrodynamics: Omnipresent modulation instability beyond simple periodicity. Phys. Rev. X 2015, 5, P. 041026.
- [3] G. Xu, A. Gelash, A. Chabchoub, V. Zakharov, B. Kibler, Ghost interaction of breathers, Frontiers in Physics, 2020, 8:608894.
- [4] A. Gelash, Formation of rogue waves from a locally perturbed condensate. Phys. Rev. E 97, 022208,2018.
- [5] A. Gelash, G. Xu, and B. Kibler, Management of breather interactions. Coming soon.

Thank you for your attention!