Exact solution to the main turbulence problem for a compressible medium and the universal -8/3 law turbulence spectrum of breaking waves

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https://doi.org/10.1063/5.005621

Physics of Fluids, vol. 33, 076108 (2021)

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Abstract

An exact analytical solution to the one-dimensional compressible Euler equations in the form of a nonlinear simple wave is obtained.

In contrast to the well-known Riemann solution, the resulting solution and the time of its collapse have an explicit dependence on the initial conditions.

The turbulence energy dissipation rate fluctuations universal spectrum is obtained $\boldsymbol{E_D(k) \propto k^{-2/3}}$ which is near the same as the experimental value $\boldsymbol{E_D(k) \propto k^{-0.67}}$ of Kholmyansky (1972) that is known for the atmospheric turbulence in the surface layer.

The exact solution for the universal spectrum $\boldsymbol{E(k) \propto k^{-8/3}}$ of turbulence energy in a compressible medium is also obtained, which is consistent with observational data for the turbulence spectrum in the solar wind and magnetosheaths of the Earth and of Saturn (F. Sahraoui, et.al PRL, 2006; R. Bandypadhyay, et.al. PRL, 2020).

1.Euler's equations

The one-dimensional Euler equation and continuity equation are represented in the form:

$$\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0 \qquad (1.1)$$

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho V)}{\partial x}=0 \qquad (1.2)$$

For the polytropic gas, additionally to (1.1), (1.2), also the following relation is considered:

$$p=p_0\left(\frac{\rho}{\rho_0}\right)^{\gamma} \qquad (1.3)$$

Assuming for the case of simple wave that V can be also represented as a function of ρ only, following relations are obtained:

$$\frac{dV}{d\rho}=\pm\frac{c}{\rho};$$

$$c^2=\frac{dp}{d\rho} \qquad (1.4)$$

The Riemann equation is represented in the next form, known also as the Hopf-Burgers (HB) equation:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0;$$

$$u=V\pm c \qquad (1.5)$$

2. Explicit solution of the Riemann problem

1.

Velocity $$\boldsymbol{V(x,t) = \int_{-\infty}^{\infty} d\xi V_0(\xi)(1 + t\frac{du_0(\xi)}{d\xi})\delta(\xi - x + tu_0(\xi)),} \quad (2.1)$$

$$\boldsymbol{u_0(\xi) = V_0(\xi) \pm c_0(\xi);}$$

Density $$\boldsymbol{\rho(x,t) = \int_{-\infty}^{\infty} d\xi \rho_0(\xi)(1 + t\frac{du_0}{d\xi})\delta(\xi - x + tu_0(\xi)); \rho_0(x) = \rho(x, t = 0)} \quad (2.2)$$

Collapse time: $$\boldsymbol{1 + t(\frac{dV_0(x)}{dx} \pm \frac{dc_0(x)}{dx}) = 0; t_0 = \frac{1}{max\left|\frac{du_0(x)}{dx}\right|}; u_0 \equiv V_0 \pm c_0} \quad (2.3)$$

For the initial velocity distributions $\boldsymbol{V_0(x) = a\,exp(-x^2/2x_0^2)}$

$$\boldsymbol{t_0 = \frac{2x_0\sqrt{e}}{a(\gamma+1)}} \quad (2.4)$$

3.1. Solution regularization by dissipation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -u\mu \qquad (3.1)$$

$$\left(\frac{\partial \rho}{\partial x}\right)^2 = e^{-2\mu t}\int_{-\infty}^{\infty} d\xi \frac{(d\rho_0/d\xi)^2\delta(\xi - x + \tau(t)u_0(\xi))}{1+\tau(t)du_0/d\xi}; \tau(t) = \frac{1-exp(-t\mu)}{\mu} \qquad (3.2)$$

Regularization condition: $\mu > \mu_{th} = \frac{1}{t_0}$ (3.3)

Regularization and predictability problem

$$\nu \Delta V \propto -\mu V;$$

$$\mu = \nu / \lambda_{\min}^2;$$

$$\lambda_{\min} \propto k_{\max}^{-1}$$

The minimum grid size $\boldsymbol{\lambda_{min}}$ in the numerical integration of the Navier-Stokes equations, which is associated with the inevitable truncations at large wave numbers $\boldsymbol{k_{max}}$.

3.2. Solution regularization by dissipation

Stochastic modeling of effective viscosity

$$\langle \boldsymbol{V}(\boldsymbol{x};\boldsymbol{t})\rangle = \int_{-\infty}^{\infty} \boldsymbol{dB}\,\frac{\boldsymbol{exp(-B^2/4t\nu)}}{\boldsymbol{2\sqrt{\pi t\nu}}}\boldsymbol{V}(\boldsymbol{x}-\boldsymbol{B},\boldsymbol{t}) \tag{3.5}$$

$$\boldsymbol{x} \to \boldsymbol{x} - \boldsymbol{B}$$

$$\boldsymbol{B}(\boldsymbol{t}) = \int_{\boldsymbol{0}}^{\boldsymbol{t}} \boldsymbol{dt_1}\widetilde{\boldsymbol{V}}(\boldsymbol{t_1}); \langle\widetilde{\boldsymbol{V}}\rangle = \boldsymbol{0}; \langle\widetilde{\boldsymbol{V}}(\boldsymbol{t})\widetilde{\boldsymbol{V}}(\boldsymbol{t_1})\rangle = \boldsymbol{2\nu\delta}(\boldsymbol{t}-\boldsymbol{t_1})$$

$\boldsymbol{u}\frac{\boldsymbol{\partial u}}{\boldsymbol{\partial x}} \to (\boldsymbol{u}+\widetilde{\boldsymbol{V}}(\boldsymbol{t}))\frac{\boldsymbol{\partial u}}{\boldsymbol{\partial x}}$ for the random Gaussian velocity field $\widetilde{\boldsymbol{V}(\boldsymbol{t})}$

Furutsu-Novikov (1964): $\left\langle\widetilde{\boldsymbol{V}}(\boldsymbol{t})\frac{\boldsymbol{\partial u}}{\boldsymbol{\partial x}}\right\rangle = -\boldsymbol{\nu}\frac{\boldsymbol{\partial^2 u}}{\boldsymbol{\partial x^2}}$

4.1. Intermittence and dissipation fluctuations

Integral kinetic energy of the turbulent flow of a compressible medium $\boldsymbol{E_C = \frac{1}{2L}\int_{-\infty}^{\infty} dx\, \rho(x;t) V^2(x;t)}$

$$\boldsymbol{\frac{1}{\rho_\infty}\frac{dE_C}{dt} = -I_D + I_P; I_D = \nu_D \Omega_2 = \frac{\nu_D}{L}\int_{-\infty}^{\infty} dx \left(\frac{\partial V(x;t)}{\partial x}\right)^2; I_P = \frac{1}{\rho_\infty L}\int_{-\infty}^{\infty} dx p(x;t)\frac{\partial V(x;t)}{\partial x}; \nu_D = (\frac{4\eta}{3} + \varsigma)/\rho_\infty}$$

For politropic medium $\boldsymbol{V(x,t) = \int_{-\infty}^{\infty} d\xi V_0(\xi)\left(1 + t\frac{(\gamma+1)}{2}\frac{dV_0}{d\xi}\right)\delta\left(\xi - x + t\left(\pm c_\infty + \frac{(\gamma+1)}{2}V_0(\xi)\right)\right)}$

$$\boldsymbol{p(x;t) = \int_{-\infty}^{\infty} d\xi p_0(\xi)\left(1 + t\frac{(\gamma+1)}{2}\frac{dV_0}{d\xi}\right)\delta\left(\xi - x + t\left(\pm c_\infty + \frac{(\gamma+1)}{2}V_0(\xi)\right)\right)}$$

$$\boldsymbol{c_0(x) = c_\infty \pm \frac{(\gamma-1)}{2}V_0(x); p_0(x) = p_\infty\left(1 \pm \frac{(\gamma-1)}{2c_\infty}V_0(x)\right)^{\frac{2\gamma}{\gamma-1}}}$$

4.2. Intermittence and dissipation fluctuations

$\boldsymbol{\varepsilon_C = -I_D}$ because for politropic medium $\boldsymbol{I_P = 0}$

Local energy dissipation rate and enstrophy: $\boldsymbol{\varepsilon(x,t) = \frac{\nu}{2}\left(\frac{\partial V(x,t)}{\partial x}\right)^2}$;

$$\boldsymbol{I_D \propto \Omega_2 \propto \frac{1}{L}\int_{-\infty}^{\infty} dx\varepsilon(x;t),}$$

Structural function: $\boldsymbol{S_D(r) = \left\langle \varepsilon(x+r;t)\varepsilon(x;t)\right\rangle - \langle\varepsilon\rangle^2}$

$\boldsymbol{(r; t = t_0) \propto r^{-1/3}}$ in the limit $\boldsymbol{r \to 0; t \to t_0}$ (4.1)

Spectrum $\boldsymbol{E_D(k) = C_D k^{-2/3}\, exp\left(-t_0 k^2 \nu\right)}$ (4.2)

In the inertial range of scales $\boldsymbol{L^{-1} << k << l_\nu = (2t_0\nu)^{-1/2}}$

$E_D \propto k^{-2/3}$ (4.3).

4.3. Intermittence and dissipation fluctuations

Up to now days, for the scaling law -5/3, corrections are introduced related to the heuristic description:

$\boldsymbol{E(k) \propto \langle \varepsilon \rangle^{2/3} k^{-\frac{5}{3}} (Lk)^{-q}}$, where $\boldsymbol{q = \frac{\mu}{9}}$ and $\boldsymbol{L}$ is the integral turbulence scale.

$\boldsymbol{\mu = \frac{\log \alpha}{\log \beta}; \frac{\lambda_1}{l_1} = \ldots = \frac{\lambda_j}{l_j} = \alpha << 1; \frac{\lambda_2}{\lambda_1} = \ldots = \frac{\lambda_j}{\lambda_{j-1}} = \beta << \alpha;\ \ 0 < \mu < 1}$, Novikov-Stewart (1964).

$\boldsymbol{E_D(k) \propto k^{-1+\mu}}$,

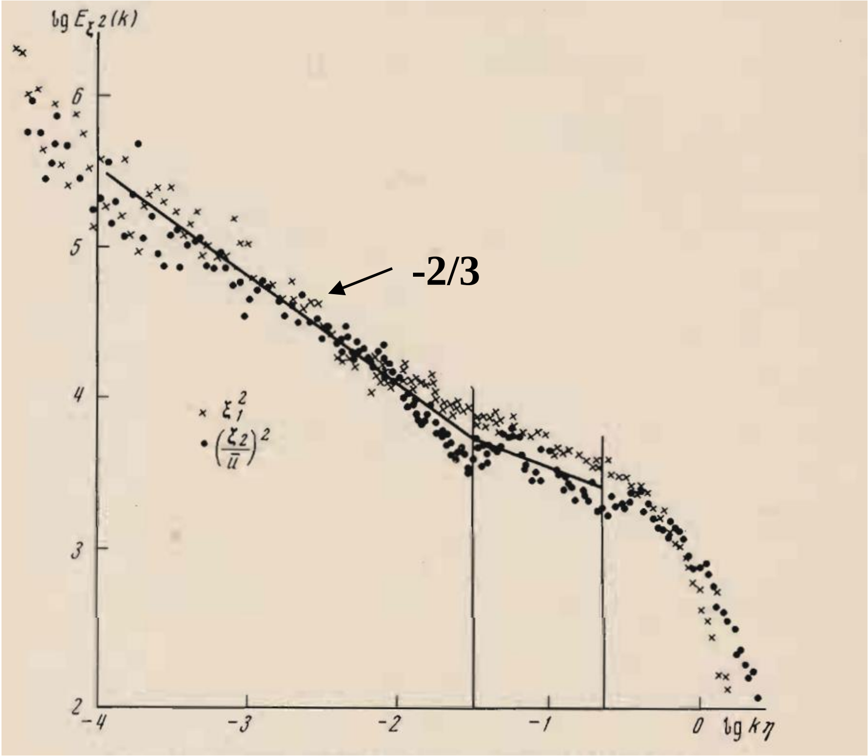
$\boldsymbol{S_D(r) = \left\langle (\varepsilon(\vec{x} + \vec{r}; t) - \varepsilon(\vec{x}; t))^2 \right\rangle \propto r^{-\mu}}$

$\boldsymbol{\mu \approx 0.38 \pm 0.05}$ Pond, Stewart (1965); $\boldsymbol{k = 0.01cm^{-1} - 10cm^{-1}}$

Gurvich , Zubkovsky 1963), (1965)

$\boldsymbol{\mu \approx 0.33}$ Kholmyansky (1972)

$\boldsymbol{\mu = 1/3}$ Exact solution (Phys. Fluids, 2021)



Spectra of the squared derivative of the wind speed

5. Turbulence energy spectrum:

$$\boldsymbol{E(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} drR(r)\, exp(-ikr) = \frac{1}{2\pi L} I(k)I^*(k)\ ;\ e = \int_0^\infty dkE(k)} \qquad (5.1)$$

$$\boldsymbol{I(k) = \int_{-\infty}^{\infty} dxV_0(x)\frac{\partial S}{\partial x}exp(ikS(x,t)) = \frac{i}{k}\int_{-\infty}^{\infty} dx\frac{dV_0}{dx}exp(ikS)\, ; I^*(k) \equiv I(-k)}$$

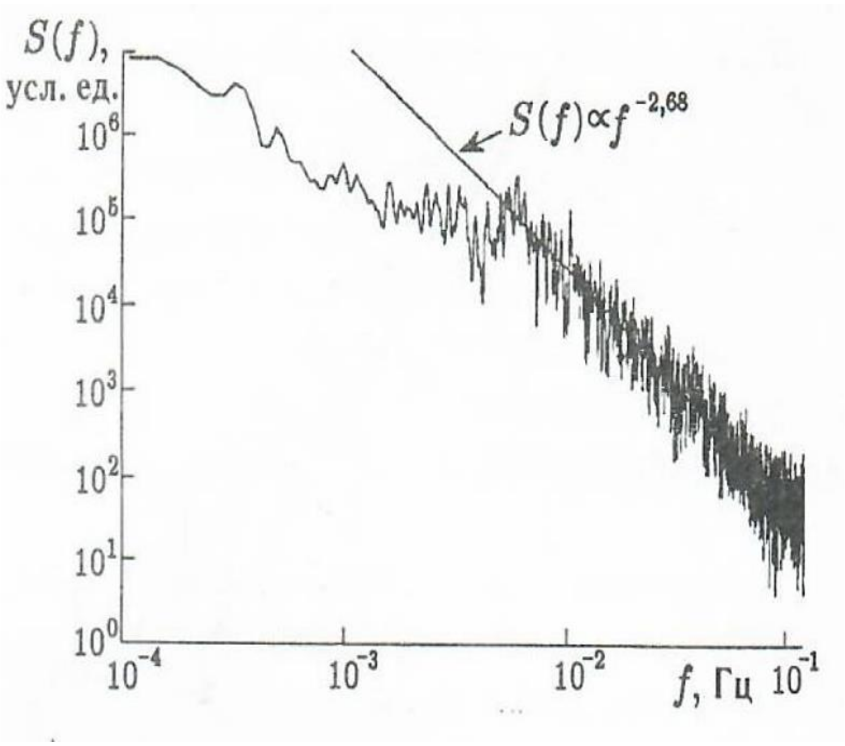
$$\boldsymbol{S(x;t) = x + \frac{(\gamma+1)t}{2}V_0(x); A(x,t) \equiv \frac{\partial S}{\partial x}}$$

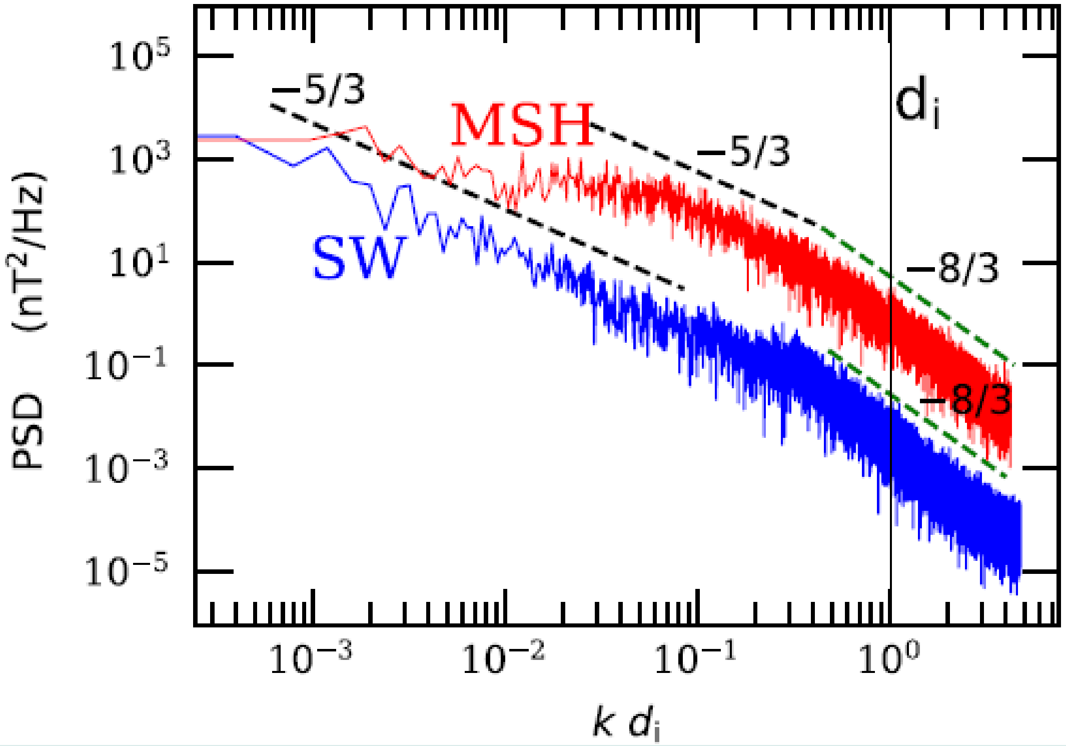
In the limit $\boldsymbol{kL >> 1}$ and $\boldsymbol{t \to t_0}$

$$\boldsymbol{E(k) = C_E k^{-8/3}} \qquad (5.2)$$

$$\boldsymbol{C_E = \frac{2^{5/3}}{L}\left(\frac{dV_0}{dx}\right)_{x=x_M}^{8/3}\left(\frac{d^3V_0}{dx^3}\right)_{x=x_M}^{-2/3}\Phi^2(0)}$$

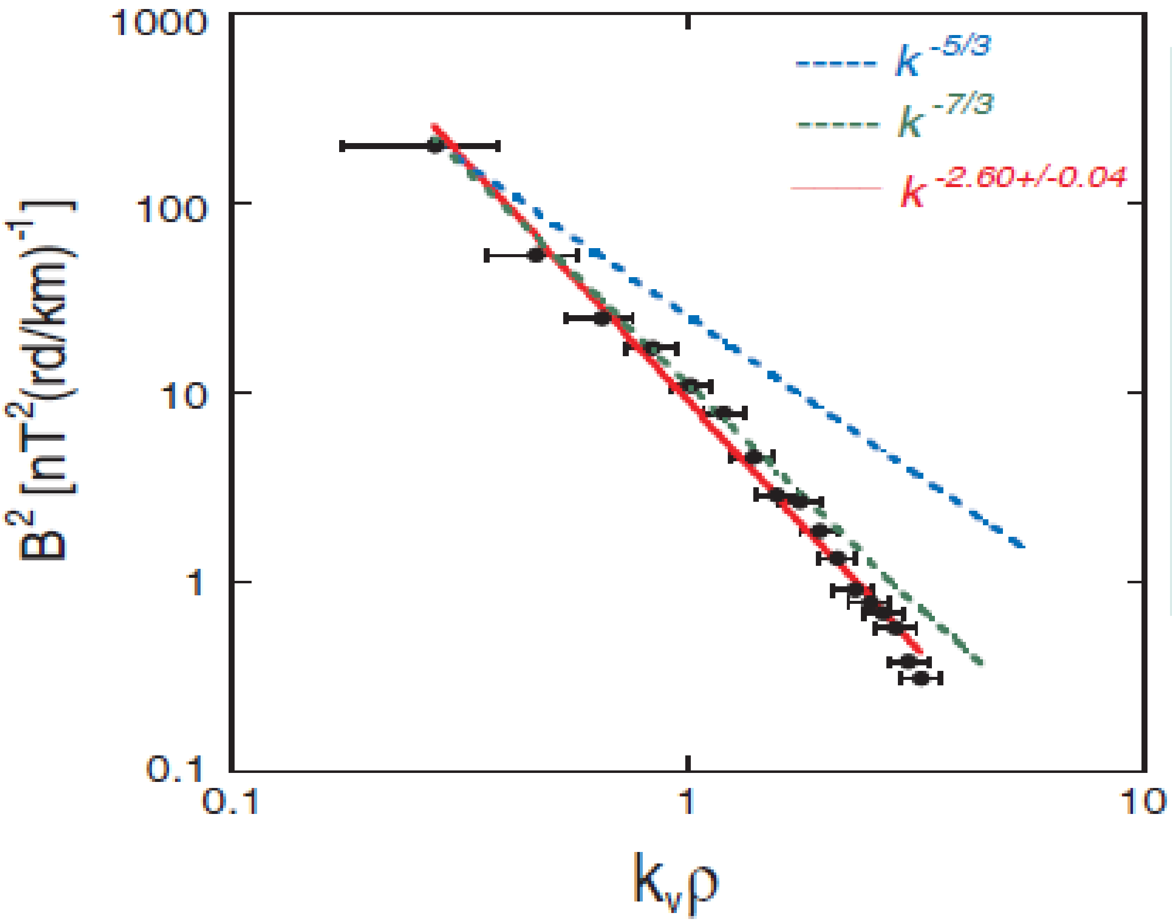
In (4.7) $\boldsymbol{\Phi(0) = \frac{1}{\sqrt{\pi}}\int_0^\infty d\,u\,cos(\frac{u^3}{3}); \Phi(z) = \sqrt{\pi}Ai(z)}$ - the Airy function and [17] (see formula (b. 8) on page 784 in [17] where $\boldsymbol{\Phi(0) = \frac{\sqrt{\pi}}{3^{2/3}\Gamma(2/3)} \approx 0.629}$).



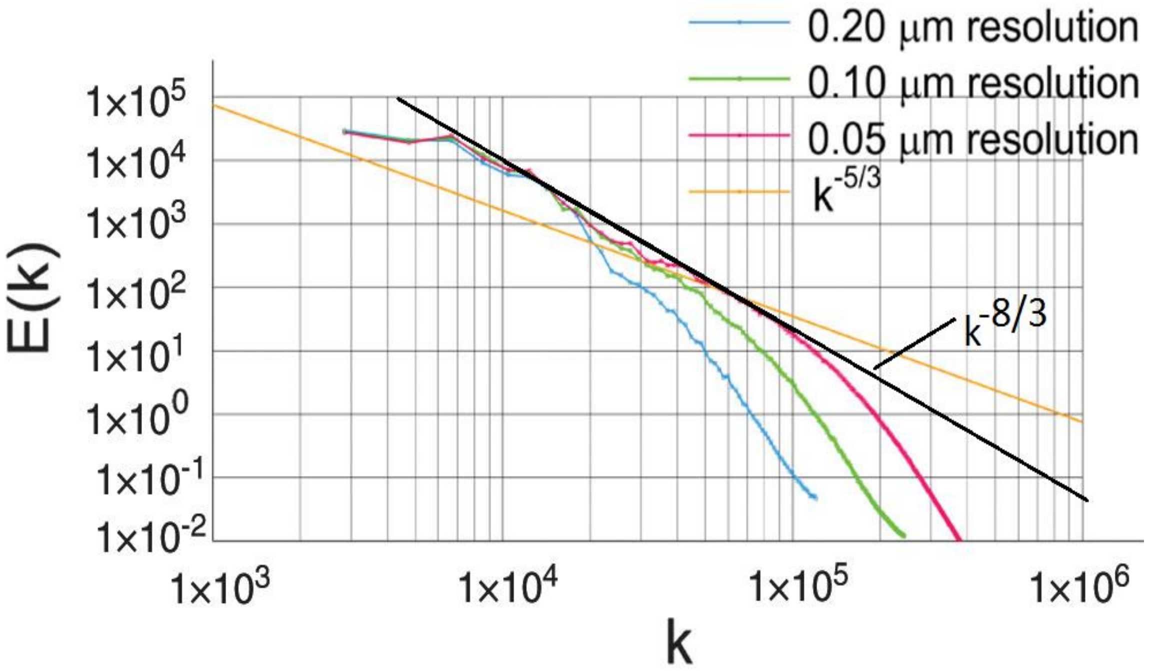


Electric field pulsation spectrum of aeroelectric structures in the surface layer of atmosphere.
Spectrum of the electric-field pulsations under fog conditions according to the data obtained at the Borok Geophysical Observatory on September 18–19, 1999 from 22:00 to 01:00 UT.-
From: S. V. Anisimov, E. A. Mareev, N. M.Shikhova, and E. M. Dmitriev, Universal spectra of electric field pulsations in the atmosphere, Geophys. Res. Lett., vol.29 (24), 2217 (2002); https://doi.org/10.1029/2002GL015765

Magnetic field turbulence spectra for the solar-wind (SW) in blue (when $\boldsymbol{d_i} = \mathbf{75}$km; $|\langle\vec{V}\rangle| = \mathbf{330}$km s) and magnetosheath (MSH) in red (when $\boldsymbol{d_i} = \mathbf{56km}$; $|\langle\vec{V}\rangle| = \mathbf{278km}$ s) interval. The solid vertical line represents $\boldsymbol{kd_i} = \mathbf{1}$ with the wave vector$\boldsymbol{k} = \boldsymbol{2\pi f}/|\langle\vec{V}\rangle|$, where$\boldsymbol{f}$is the frequency [12] (see Fig.1 in R. Bandyopadhyay, et.al. , PRL 2020).

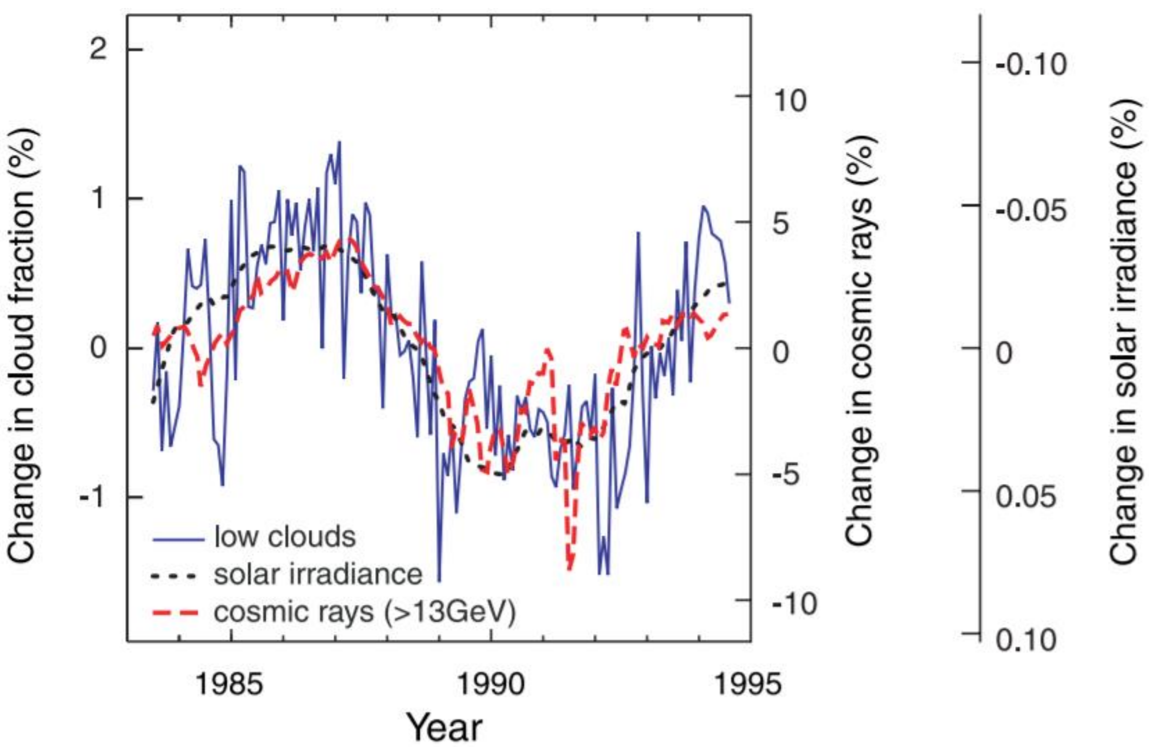


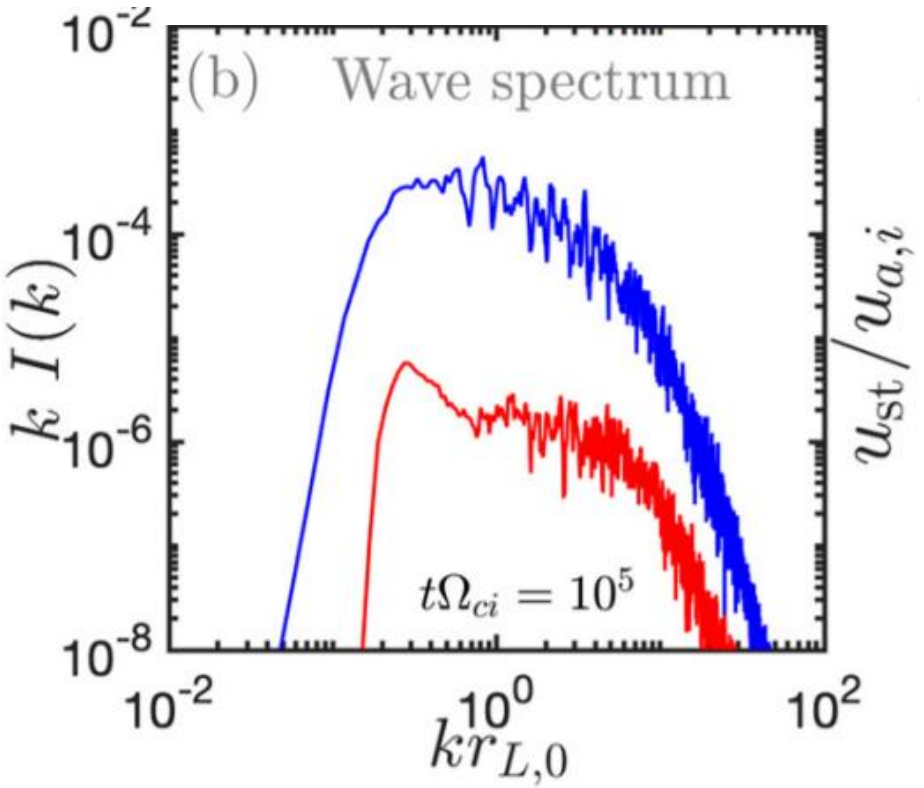
Magnetic field turbulence spectrum for the magnetosheath (when proton Larmor radius $\boldsymbol{\rho = 75}$km). The red line is a direct fit revealing a power law $\boldsymbol{k^{-2.6}}$. Two other power laws are plotted for comparison: $\boldsymbol{k^{-7/3}}$ (green) and $\boldsymbol{k^{-5/3}}$ (blue) (see Fig.6 in F. Sahraoui, et.al, PRL, 2006).F. Sahraoui, G. Belmont, L. Rezeau, and N. Cornilleau-Wehrlin, Anisotropic turbulent spectra in the terrestrial magnetosheath as seen by the Claster spacecraft, Phys. Rev. Lett., 96, 075002 (2006); www.doi.org/10.1103/PhysRevLett.96.075002



Turbulence kinetic energy spectra on the stagnation stage of fusion implosion at t=1.71 ns for three different simulations resolutions are taken from Fig.3 in Thomas V. A. , and Kares R. J., Drive asymmetry and the origin of turbulence in a inertial confinement fusion implosion Phys. Rev. Lett., 109, 075004 (2012)

The possibility of arise the helical turbulence with the chiral symmetry breaking on the stages before stagnation is stated on this base. A possible reason for this is a violation of spherical symmetry during the fusion implosion due to the occurrence of rotation of the medium behind the front of a converging spherical shock wave by the instability mechanism. Indeed, on the basis of numerical simulation Galtier S., and V. David, Inertial/kinetic-Alfvén wave turbulence: A twin problem in the limit of local interactions,Phys. Rev. Fluids, 5, 044603 (2020) is stated that in the perpendicular direction to the direction of rotation, the inertial waves turbulence has an anisotropic one-dimensional scaling law−8/3 as in the anisotropic turbulence spectrum in the space plasma.





Variation of low-altitude cloud cover, cosmic rays,
and total solar irradiance between 1984 and 1994.
The cosmic ray intensity is from Huancayo observatory, Hawaii.

Results of PIC-MHD simulations of the resonant cosmic ray SI, including the effect of ion–neutral damping. The parameters of these simulations are as follows: initial streaming speed of CRs is ud=ua ¼ 10, density ratio is nCR=nbg ¼ 104 , and the typical energy of CRs is E0 GeV.

Panel shows the magnetic wave spectrum at the early saturated phase of the instability

Panel compares the case when there is no damping (blue lines) and with moderate ion–neutral damping using red lines (damping rate in is 0.5 of the maximum growth rate of the instability, Cmax).

Conclusions

Thus the exact closed-form explicit analytical solution to the Riemann problem for the Euler one-dimensional hydrodynamics equations is obtained.

The regularization by dissipation factors is determined for that solution for unlimited time, which gives unexpected positive resolution for the generalization of the Clay problem (www.claymath.org) to the fluid and gas dynamics in the compressible case.

On the base of explicit analytical form of the Riemann solution the explicit representation for the shock wave arising time is obtained for arbitrary initial conditions of the simple wave.

Closed explicit analytical representations are obtained for one-point and two-point moments of hydrodynamic fields and for the energy spectrum, which gives an example of solving the turbulence problem based on the exact solution of one-dimensional Euler equations for a compressible medium.

In this case, the turbulence spectrum power-law obtained from the exact solution of the Euler equation corresponds to the parameters of the turbulent spectra observed in the Earth's and Saturn's magnetospheres, as well as for the Solar Wind and for the turbulence arising during the fusion implosion.

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