Influence of boundary on folding in inviscid fluids

E.A. Kuznetsov [1,2,3] and E.A. Mikhailov[1,3,4]

[1] - P.N. Lebedev Physical Institute, Moscow, Russia

[2] - L.D. Landau Institute for Theoretical Physics, Chernogolovka, Moscow region, Russia

[3] - Skoltech, Skolkovo, Moscow region, Russia

[4] - Physics Dept, Moscow State University

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roduction & Motivation: Collapse and the Kolmogorov-Obukhov theory

- According to the Kolmogorov-Obukhov theory (1941) velocity fluctuations at spatial scales l from the inertial range obey the power-law $\langle|\delta v|\rangle \propto \varepsilon^{1/3} l^{1/3}$, where ε is the mean energy flux from large to small scales. This formula is easily obtained from the dimensional analysis.
- Similarly, fluctuations for the vorticity field $\omega = \nabla \times \mathbf{v}$ diverge at small scales as $\langle|\delta\omega|\rangle \propto \varepsilon^{1/3} l^{-2/3}$, while the time of energy transfer from the energy-contained scale l_E to the viscous ones is finite and estimated as $T \sim l_E^{2/3} \varepsilon^{-1/3}$.
- These two relations allow to link the Kolmogorov spectrum formation with the blowup in the vorticity field (collapse).

Introduction & Motivation

- The question whether finite time singularities develop in inertial scales (in fact, in ideal fluids) is still open question, in spite of certain progress in both numerical and analytical studies.
- Up to now, the question about blow-up existence for ideal fluids within the 3D Euler remains controversial. In our numerics (Agafontsev, Kuznetsov, Mailybaev 2015, 2017, 2019, 2022) for periodical boxes we have observed formation of high-vorticity structures of the pancake type with exponential growth of ω but without any tendency to blowup. Such increasing is connected with the vorticity compressibility. The latter follows from the vorticity ω frozen-in-fluids.

Introduction & Motivation

- However, for flows of ideal fluids in the presence of rigid boundaries recent findings, both analytical and numerical, demonstrate blow-up behavior. For two-dimensional planar flows in the region with non-smooth boundaries Kiselev and Zlatos (2015) proved blow-up existence.
- In 2014, 2015 Luo and Hou in numerical experiments for axi-symmetrical flows with swirl inside the cylinder of constant radius observed appearance of collapse just on the boundary. It was a challenge why boundaries play so important role in formation of singularities.

Introduction & Motivation

- In 1985 E and Engquist reported some rigorous results about blow-up existence for inviscid Prandtl equation for some initial data.
- It is worth noting that before, in 1980, the blowup appearance in the Prandtl equation was observed in the numerical simulations by Van Dommelen and Shen.
- In 2003 Hong and Hunter investigated this problem for both viscous and inviscid Prandtl equation for zeroth pressure gradient. In particular, in the inviscid case they noticed that singularity can form on the wall.

Introduction & Motivation

- In this talk we show that flat boundary itself introduces some element of compressibility into flow which from our point of view can be considered as a reason of the singularity formation on the boundary. We will consider the 2D and 3D inviscid Prandtl equations which describes the dynamics of the boundary layer, and demonstrate that singularity is formed for the velocity gradient on the wall. . This process is nothing more than breaking (or folding for 2D Euler) phenomenon which is well known in gas dynamics since the classical works of famous Riemann.

Basic equations and mixed Lagrangian-Eulerian description

The inviscid Prandtl equation for 2D flows is written for the velocity component parallel to the blowing plane $y = 0$:

$$u_t + uu_x + vu_y = -P_x, \;\; u_x + v_y = 0$$

with the following boundary conditions:
$v|_{y=0} = 0, \;\; \lim_{y \to \infty} u(x, y, t) = U(x).$
NOTE: The Prandtl equation assumes that the along surface scale L much larger the boundary layer thickness h: $L \gg h$. Hence one can see from incompressibility condition that $u/L \approx v/h$, i.e. $u \gg v$. As a result, the pressure $P = P(x)$. It gives the Prandtl equations

Basic equations and mixed Lagrangian-Eulerian description

Within the Prandtl approximation for inviscid flows it is possible to introduce the vorticity as

$$\omega = -\frac{\partial u}{\partial y}$$

which satisfies the equation of the same form as for the 2D Euler fluids:

$$\omega_t + u\omega_x + v\omega_y = 0.$$

Thus, ω is the Lagrangian invariant. By this reason, its values will be bounded at all $t > 0$. However, for another components of the velocity gradient such restrictions are absent. As we will see below, u_x as well as v_y can take arbitrary values, in particular, infinite ones.

Basic equations and mixed Lagrangian-Eulerian description

For $P_x = 0$, u is a Lagrangian invariant. Let n be some Lagrangian quantity (advected by the fluid), obeys the equation

$$n_t + un_x + vn_y = 0, \;\; u_x + v_y = 0.$$

For its solution $n = n(x, y, t)$, define inverse function $y = y(x, n, t)$. In this case we have new independent Lagrangian variable n and old Eulerian coordinate x (note, for the Prandtl equation such transformation was introduced first time by Crocco). Transition to this description is the mixed Eulerian-Lagrangian one which represents non-complete Legendre transformation. Fixing n in $y = y(n, x, t)$ yields the n-level line and therefore this transform is the transition to the movable *curvilinear* system of coordinates.

Basic equations and mixed Lagrangian-Eulerian description

Then we find how derivatives with respect to variables (x, y, t) (the l.h.s) and derivatives relative to (x, n, t) (the r.h.s) are connected with each other:

$$\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{1}{y_n}\left[f_t y_n - f_n y_t\right], \\
\frac{\partial f}{\partial x} &= \frac{1}{y_n}\left[f_x y_n - f_n y_x\right], \\
\frac{\partial f}{\partial y} &= \frac{f_n}{y_n}.
\end{aligned}$$

Substitution of these transforms into the equation of motion for n gives the kinematic condition, well known for free-surface hydrodynamics:

$$y_t + u y_x = v.$$

Basic equations and mixed Lagrangian-Eulerian description

Introduce streamfunction ψ so that $u = \psi_y,\ v = -\psi_x$. By means of formulas for derivatives these relations read as

$$u = \frac{1}{y_n}\psi_n,\ v = -\psi_x + \frac{y_x}{y_n}\psi_n.$$

Substitution of these formulas into the equation for y results in the linear relation between y and ψ:

$$y_t = -\psi_x.$$

Note, that in this equation all derivatives are taken for fixed n. This equation can be easily resolved by introducing the generating function $\theta(x, n, t)$:

$$y = \theta_x,\ \psi = -\theta_t.$$

To find $\theta(x, n, t)$ one needs to know dynamics of the velocity.

Solution to the inviscid Prandtl equation

Consider first $P_x = 0$. In this case for the inviscid Prandtl equation we have Eq.

$$u = \frac{1}{y_u} \psi_u,$$

which after substitution of θ transforms into

$$\frac{\partial \theta_u}{\partial t} + u \frac{\partial \theta_u}{\partial x} = 0.$$

This equation evidently has the following solution:

$$\theta_u = F(x - ut, u)$$

where F is an arbitrary smooth function determined from the boundary-initial conditions. Integration with respect to u yields

$$\theta = \int_{f(x,t)}^{u} F(x - zt, z) dz + g(x, t).$$

Solution to the inviscid Prandtl equation

Here $f(x,t)$ and $g(x,t)$ are another arbitrary functions to be defined from the B-I conditions.

It is worth noting that at $y=0$ and $P_x=0$ the inviscid Prandtl equation is nothing more than the Hopf equation

$$u_t + uu_x = 0,$$

which solution is written in the following implicit form (simple Riemann wave)

$$u = u_0(a), \;\; x = a + u_0(a)t$$

or

$$u = u_0(x - ut).$$

This means that on the boundary we have breaking, i.e. the formation of singularity in a finite time.

Solution to the inviscid Prandtl equation

Breaking happens when the derivative

$$\frac{\partial u}{\partial x} = \frac{u_0'(a)}{1 + u_0'(a)t}$$

at some point $x = x_*$ first time, $t = t_*$, becomes infinite. It is evident that $t_* = \min_a \left[-1/u_0'(a)\right]$. Then it is possible to establish that the general solution is matched with the boundary conditions at $y = 0$ if one puts

$$f(x,t) = u(x,0,t)$$

(this is solution of the Hopf equation) and $g(x,t) = 0$ so that

$$\begin{aligned} y &= \int_{f(x,t)}^{u} \frac{\partial}{\partial x} F[x - zt, z] dz \\ \psi &= -\int_{f(x,t)}^{u} \frac{\partial}{\partial t} F[x - zt, z] dz. \end{aligned}$$

Solution to the inviscid Prandtl equation

Near the breaking point,

$$u_x \simeq -\frac{1}{\tau + \beta(\Delta a)^2}$$

where $\tau = t_* - t,\ \Delta a = a - a_*$.

Thus, this dependence demonstrates a self-similar compression, $\Delta a \propto \tau^{1/2}$. The denominator, up to the constant multiplier C, coincides with the Jacobian, $J = \partial x/\partial a = C\,(\tau + \beta a^2)$, where we put $a_* = 0$. Integration of this equation gives the cubic dependence: $x = C\,(\tau a + \beta a^3/3)$. Thus, in the physical space we get more rapid compression than in the a-space : $x \propto \tau^{3/2}$.

Solution to the inviscid Prandtl equation

For $\beta a^2 \gg \tau$, the Jacobian becomes time-independent, $J \sim x^{2/3}$. Hence, as $\tau \to 0$ we arrive at the singularity for u_x, $u_x \propto x^{-2/3}$. Any time changes of u_x happen at the narrowing region in the a-space, $a \propto \tau^{1/2}$, or equivalently at $x \propto \tau^{3/2}$. It results in the following self-similar asymptotics,

$$u_x \simeq \frac{1}{\tau} F(\xi), \ \xi = \frac{x}{\tau^{3/2}}$$

where function $F(\xi)$ as $\xi \to \infty$ is $\sim \xi^{-2/3}$. Hence we have the power law:

$$\max |u_x| \propto \ell^{-2/3}.$$

This is a general asymptotics for folding, independently whether the singularity happens in finite or infinite time.

Solution to the inviscid Prandtl equation

For arbitrary dependence $P(x)$ a solution is found from integration of ODEs:

$$\frac{d}{dt}u = -P_x, \;\; \frac{d}{dt}x = u,$$

which are equivalent to the Newton equation: $\ddot{x} = -P_x$. The first integral (energy) $E(a) = \dot{x}^2/2 + P(x) = u_0^2(a)/2 + P(a)$, allows to define the mapping $x = x(a, t)$. The breaking time t_* is found as a minimal root $T(> 0)$ for equation $J(a, T) = 0$, where $t_* = \min_a T(a)$ and $J = \partial x/\partial a$.

Behavior for the vorticity gradients on the boundary

Now calculate how ω behaves at the breaking point. Remind, $\omega = -u_y$ is the Lagrangian invariant.

For simplicity consider the pressureless case. Differentiation of the vorticity equation with respect to x and then putting $y = 0$ where $v = 0$ and $v_x = 0$ yield the following

$$\frac{\partial \omega_x}{\partial t} + u \frac{\partial}{\partial x} \omega_x = -u_x \omega_x.$$

The equations for characteristics are $dx/dt = u(x,t), \;\; d\omega_x/dt = -u_x \omega_x$. Substitution of $u_x \simeq (t - t_*)^{-1}$ at the breaking point gives the same singular behavior for ω_x there:

$$\omega_x \simeq \frac{A}{t - t_*}.$$

Solution to the 2D inviscid Prandtl equation

Concluding this part, note that singularities for the velocity gradient on the boundary is a result of collision of two counter-propagating slipping flows. In the first simulations (Dommelen and Shen, 1980; Hong and Hunter, 2003) this interaction was shown to lead to the formation of jets in perpendicular to the boundary direction. Breaking (as a folding happening in a finite time) for the slipping flows in the 2D Prandtl equation becomes possible because the pressure gradient normal to the boundary can not prevent the formation of jets.

Breaking in the 3D inviscid Prandtl equations

- The 3D inviscid Prandtl equations have the form

$$\mathbf{u}_t + (\mathbf{u}\nabla)\mathbf{u} + v\mathbf{u}_z = -\nabla P(\mathbf{r}), \ (\nabla\mathbf{u}) + v_z = 0$$

where $\mathbf{r} = (x, y)$ and $\mathbf{u}$ are, respectively, coordinates and velocity components parallel to the wall, $\nabla = (\partial_x, \partial_y)$, v is the normal ($\| z$) velocity component.

- Hence for slipping boundary conditions we arrive at the 2D Hopf equation

$$\mathbf{u}_t + (\mathbf{u}\nabla)\mathbf{u} = -\nabla P(\mathbf{r})$$

which also gives breaking.

Breaking in the 3D inviscid Prandtl equations

Consider for simplicity the case $P(\mathbf{r}) = \text{const}$. Then the velocity gradient $U_{ij} = \partial u_i / \partial x_j$ satisfies the following matrix equation

$$\frac{dU}{dt} = -U^2$$

which solution has the form

$$U = U_0(a)(1 + U_0(a)t)^{-1}$$

where $U_0(a)$ and a are the initial values of U and positions of fluid particles.

Breaking in the 3D inviscid Prandtl equations

Expanding $U_0(a)$ through the projectors P_α yields

$$U = \sum_\alpha \frac{\lambda_\alpha P_\alpha}{1 + \lambda_\alpha t}$$

Hence it is seen that the breaking time

$$t_0 = \min_{\alpha, a} (-\lambda_\alpha)^{-1}.$$

Near $t = t_0$

$$U \propto (t_0 - t)^{-1}$$

with the main contribution originating from the eigen value corresponding to t_0.

Breaking in the 3D inviscid Prandtl equations

This gives simultaneous singularities for both symmetric part (stress tensor)

$$S = 1/2(U + U^T)$$

and antisymmetric part (vorticity)

$$\Omega = U - U^T.$$

Singularities for both parts have the cusp form, like in the 1D case. Note that in this case, unlike 1D where the breaking criterion is written as $u_0' < 0$, we have a few restrictions on λ which are defined from quadratic equation. The first condition is that eigen values λ should be real. Secondly, λ has to take negative values.

Breaking in the 3D inviscid Prandtl equations

As we see breaking of the slipping flows in 2D Prandtl and 2D Euler is accompanied by the appearance of jets in the perpendicular direction to the slipping boundary. The same situation takes place in 3D (at least, for the inviscid Prandtl equations). From another side, breaking (or folding happening in a finite time) in the inviscid Prandtl approximation for the general initial conditions should produce growth of the perpendicular to the slipping boundary vorticity. Combination of these both factors gives an indication for understanding a mechanism of tornado generation.

Conclusion

- We have developed a new concept of the formation of big velocity gradients with the blow-up behavior or with the exponential in time increase for the slipping flows in incompressible inviscid fluids. These processes develop as a folding due to compressible character of the slipping flows.
- For the 2D inviscid Prandtl equation we have developed the mixed Lagrangian – Eulerian description based on the Crocco transformation.
- Application of this description to the inviscid Prandtl equation allows to construct its general solution written in the implicit form.

THANKS FOR YOUR ATTENTION