Two-Dimensional Turbulence:

Fluctuation of Energy Flux

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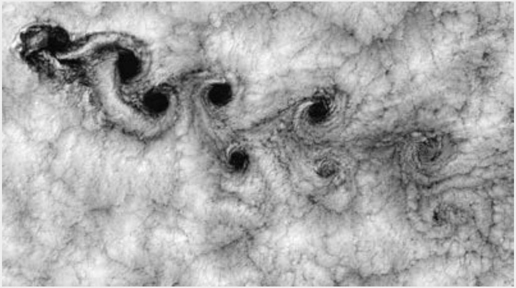


$$\vec{v}_{(3)} = (\vec{v}_{(2)}(z), 0)$$
$$\vec{v}_{(2)}|_{z=0} = 0, \qquad \vec{v}_{(2)}|_{z=h} = \vec{v}$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}, \nabla)\vec{v} = -\frac{\nabla p}{\rho} - \alpha\vec{v} + \nu\Delta\vec{v} + \vec{f} \quad \text{(2d Navier–Stokes eq.)}$$

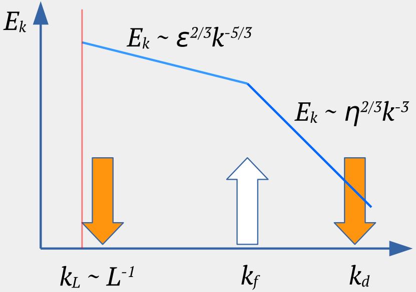
$$E = \frac{1}{2S}\int v^2 \mathrm{d}S$$

$$E_k = \frac{1}{2}\sum_{|\mathbf{k}|=k} |\vec{v}_{\mathbf{k}}|^2$$

$$\gamma_k = \alpha + \nu k^2$$

$$L_\alpha \sim \epsilon^{1/2}\alpha^{-3/2}, \; U_L \sim \epsilon^{1/2}\alpha^{-1/2}$$

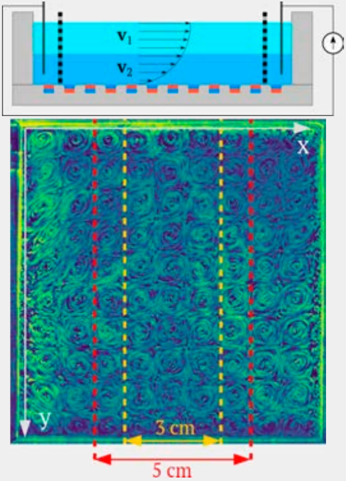
$$\epsilon(t) = S^{-1}\int \vec{f}\cdot\vec{v}(t)\,\mathrm{d}x\,\mathrm{d}y$$



Experimental Setup

$$f_x = 0$$
$$f_y = f_0 \cos\frac{\pi x}{a} \sin\frac{\pi y}{a}$$
$$a = 1\,cm$$
$$k_f = 4.4\,cm^{-1}$$

$$B(z) = \frac{B_0\ R^2}{(R^2+z^2)^{3/2}}$$
$$B_0 \approx 1.1\,T,\ 2R = 0.5\,cm$$
$$j = \frac{0.5\ A}{10\,cm \times 5\,mm} = 0.1\ A/cm^2$$
$$f_0 = \frac{j\,B}{\rho} = \frac{10^3\ A/m^2 \times 0.1\,T}{1.1\times 10^3\ kg/m^3}$$
$$f_0 \sim 10\,cm/s^2$$



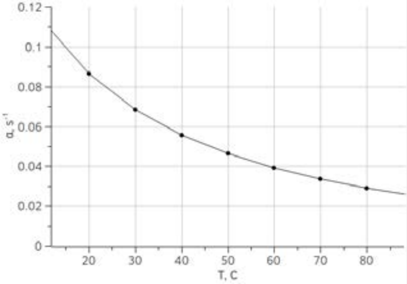
$$\epsilon(t) = S^{-1}\int f_0 \cos(k_f x)\sin(k_f y)\, v_y(t,x,y)\,\mathrm{d}x\,\mathrm{d}y > 0 \Rightarrow$$

$$\Rightarrow \langle v_{\pm k_f}\rangle \neq 0,\ \langle v(t,x,y)\rangle \neq 0$$

Cell $L_x \times L_y$: 10×10 cm, 5×10 cm and 3×10 cm

Upper: $H_2O + 15\% \, KNO_3$
$h = 0.5 - 0.7 \, cm$,
$\rho = 1.11 \, g/cm^3$,
$\nu \simeq 0.0085 \, cm^2/s$

Lower: $C_{10}F_{18}$
$h = 0.3 \, cm$,
$\rho = 1.98 \, g/cm^3$,
$\nu = 0.028 \, cm^2/s$



$$\tan \sqrt{\frac{\alpha}{\nu_t}} h_t \, \tan \sqrt{\frac{\alpha}{\nu_b}} h_b = \frac{\rho_b}{\rho_t} \sqrt{\frac{\nu_b}{\nu_t}}$$

FC ($3 \, mm$) & Electrolyte ($5 \, mm$) — $\alpha = 0.068 \, s^{-1}$, $\tau = 7.4 \, s$
Electrolyte ($8 \, mm$) — $\alpha = 0.033 \, s^{-1}$, $\tau = 15.3 \, s$
FC ($3 \, mm$) & Electrolyte ($7 \, mm$) — $\alpha = 0.037 \, s^{-1}$, $\tau = 13.6 \, s$

Single Layer of Electrolyte

Cell dimensions $L_x \times L_y$: 9×10 cm and 5×10 cm

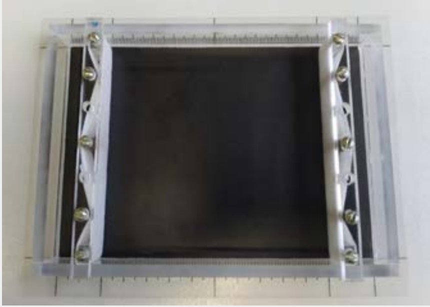
Fluid: $H_2O + 17\%\, KNO_3$

$h = [0.29, \ldots, 1.24]\, cm$

$\rho \simeq 1.12\, g/cm^3$

$\nu \simeq 0.008\, cm^2/s$

$T = 25\, ^oC$

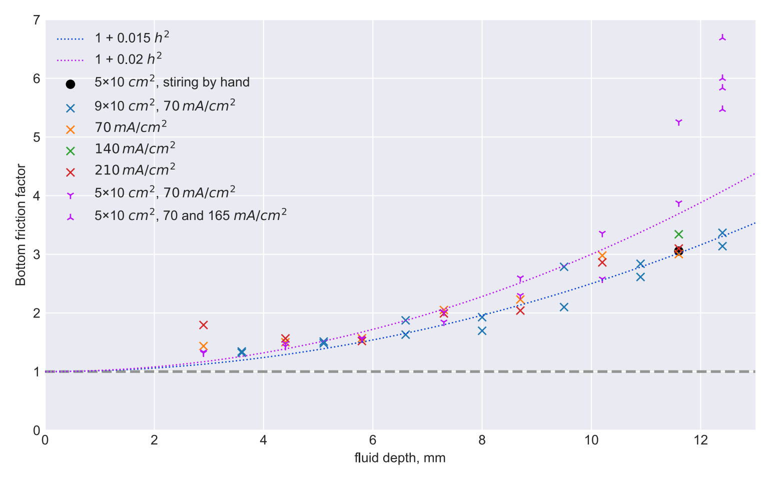


$$\alpha = \frac{\pi^2}{4} \frac{\nu}{h^2} \quad (Re \lesssim 1) \qquad \tau = \frac{1}{2\alpha}$$

Water ($T = 25\, ^oC$, $h = 0.5\, cm$): $\alpha = 0.089\, s^{-1}$, $\tau = 5.6\, s$

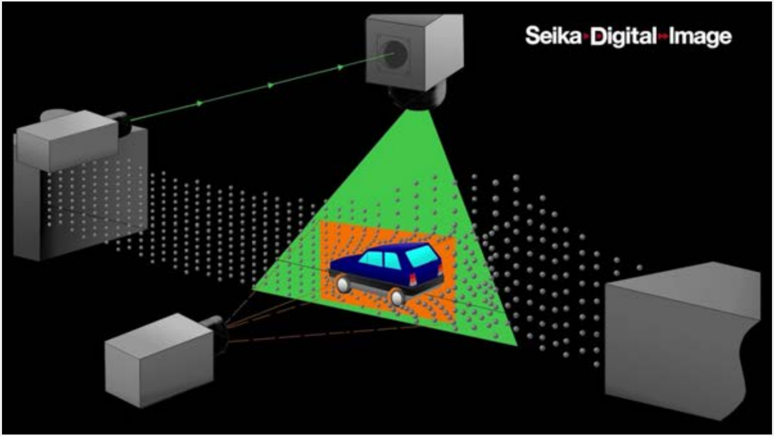
Electrolyte: $\alpha = 0.079\, s^{-1}$, $\tau = 6.3\, s$

Single Layer of Electrolyte. Dissipation factor



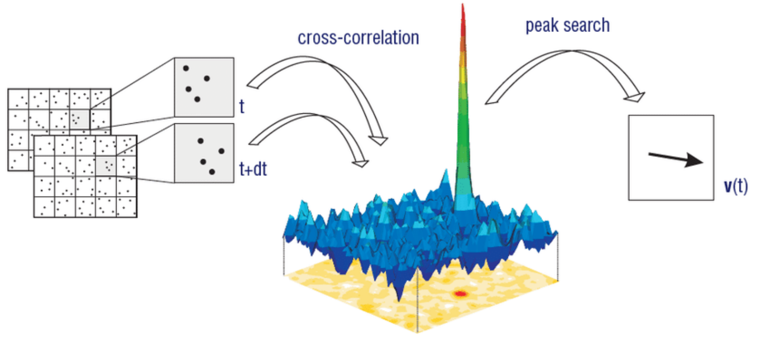
Bottom friction factor = $\frac{\alpha_{exp}}{\alpha_0}$ $\qquad \alpha_0 = \frac{\pi^2}{4}\frac{\nu}{h^2}$

Particle Image Velocimetry



Credit: SEIKA Digital Image Corporation

Particle Image Velocimetry



B. Wieneke, PIV Uncertainty Quantification and Beyond,
https://doi.org/10.13140/RG.2.2.26244.42886

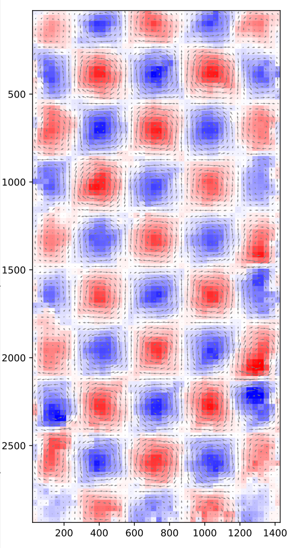
FC&Electrolyte

grid	$\sim 10^2 \times 10^2$	
$\Delta x, \Delta y$	$\approx 1\, mm$	
Δt	$0.02\, s$	

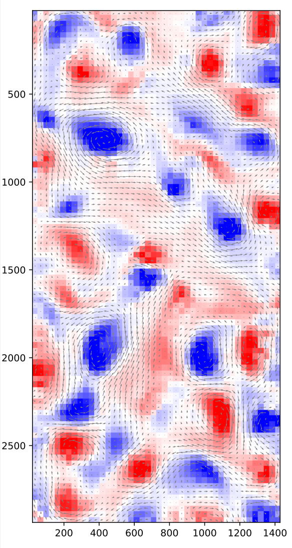
$j = 210\, mA/cm^2$

t	1	$28\, s$
$\|\|v\|\|_2$	0.6	$1.2\, cm/s$
$\|\|\omega\|\|_2$	2.5	$4.8\, s^{-1}$
$\frac{vh}{\nu}$	10^2	

$\|\cdot\|_2 = S^{-1} \int |\cdot|^2 \mathrm{d}S$

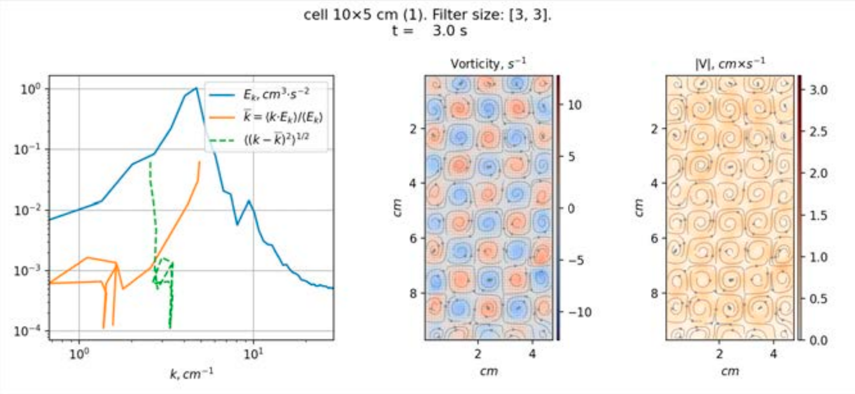


$t - t_{on} = 1\, s$



$t - t_{on} = 28\, s$

FC&Electrolyte. Cell: 5×10 cm. $t - t_{on} = 1\, s$

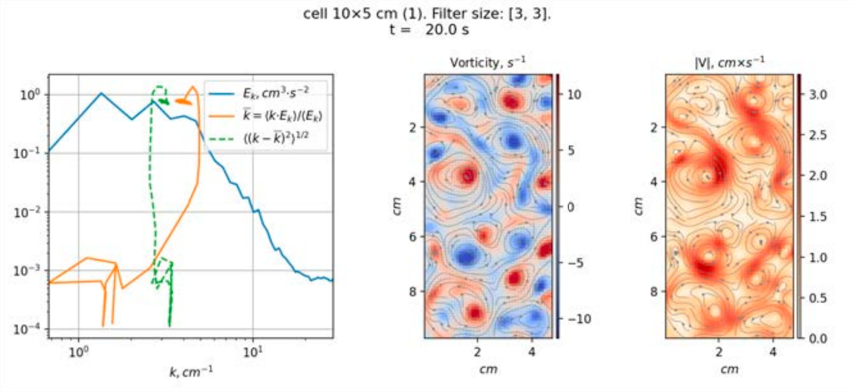


$$t - t_{on} = 1\, s$$

$$\bar{k}(t) = \langle k \rangle \qquad \Delta k(t) = \langle (k - \bar{k}) \rangle^{1/2} \qquad E(t)$$

$$\langle \xi \rangle = \frac{\int \xi E_k(t) \mathrm{d}k}{\int E_k(t) \mathrm{d}k}$$

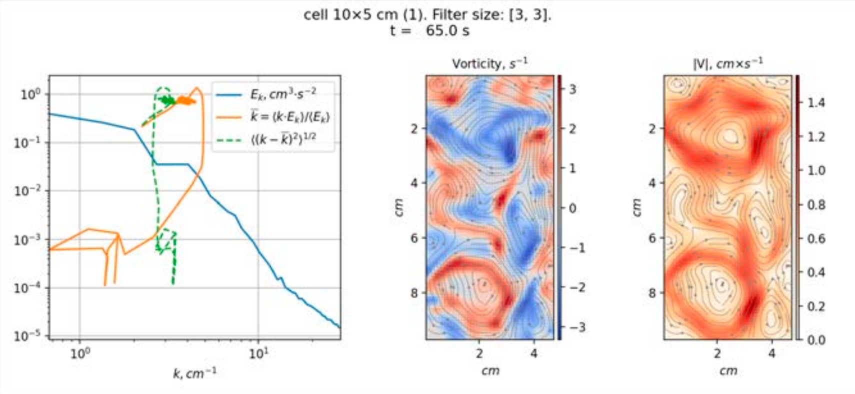
FC&Electrolyte. Cell: 5×10 cm. $t - t_{on} = 18\,s$



$$t - t_{on} = 18\,s$$

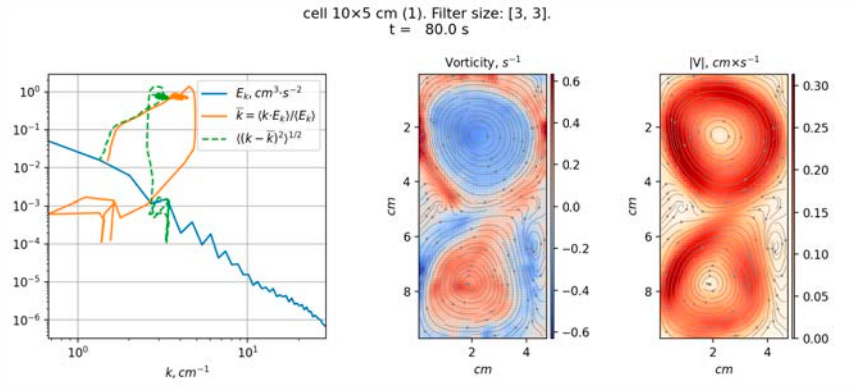
$$Re(k_f) \simeq 300,\ Re(k_L) \simeq 10^3$$

FC&Electrolyte. Cell: 5×10 cm. $t - t_{off} = 3\,s$



$$t - t_{off} = 3\,s$$

FC&Electrolyte. Cell: 5×10 cm. $t - t_{off} = 18\,s$

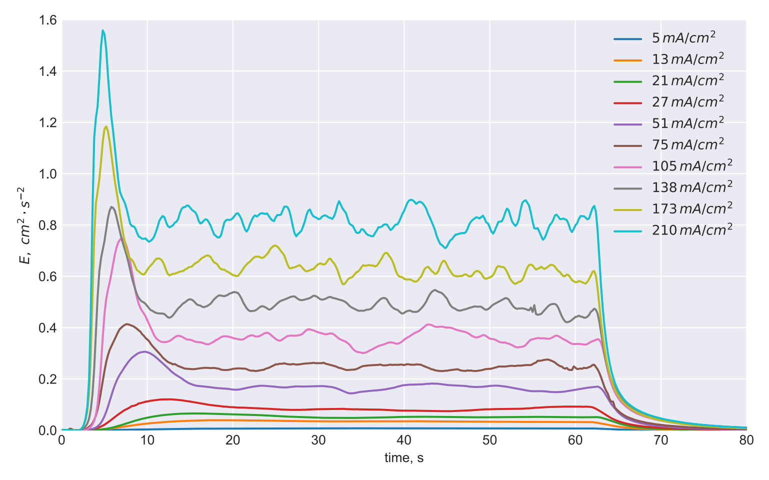


$$t - t_{off} = 18\,s$$

$$k_L = 2\pi/L_x \approx 1.3\,cm^{-1}$$

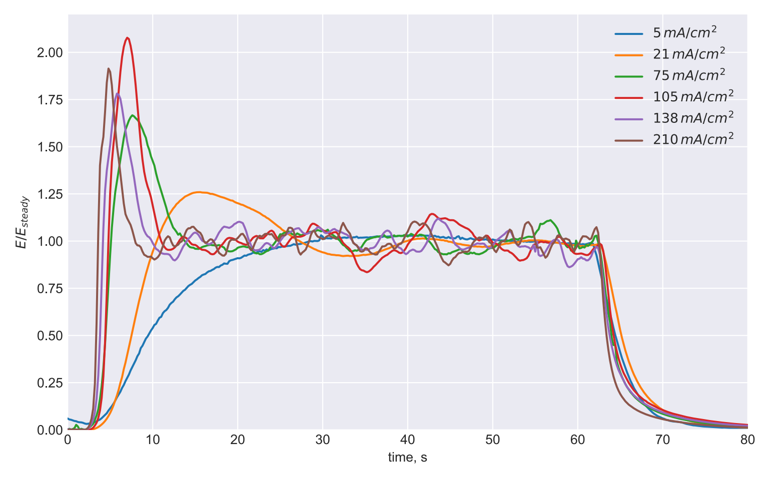
$$\bar{k} \approx 1.6\,cm^{-1}$$

FC&Electrolyte. Total Energy



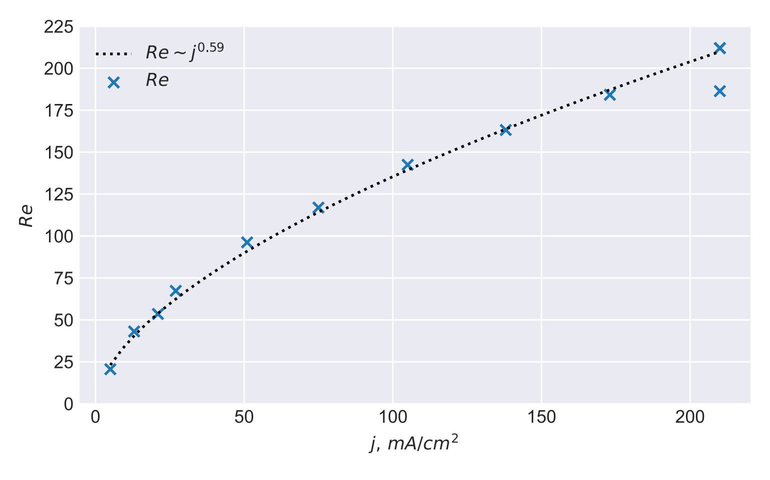
$$E(t) = (2S)^{-1} \int v^2 \, \mathrm{d}x \, \mathrm{d}y \qquad \epsilon(t) = S^{-1} \int \vec{f} \cdot \vec{v}(t) \, \mathrm{d}x \, \mathrm{d}y$$

FC&Electrolyte. Total Energy



$$E_{steady} = T^{-1} \int E(t)\,\mathrm{d}t, \quad t = [20, 60]\ s$$

FC&Electrolyte. Reynolds number



$$Re = \frac{||v||_2 \; l_f}{\nu} \sim j^{0.6}$$

Velocity Decomposition via Space Filtering

$$v(\mathbf{r}, t) = \bar{v}(\mathbf{r}, t) + \tilde{v}(\mathbf{r}, t)$$

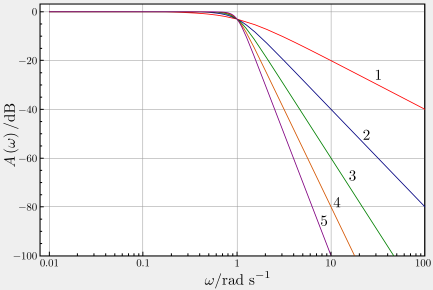
$$v_{\mathbf{k}}(t) = \sum_{\mathbf{r}} v(\mathbf{r}, t) e^{i(\mathbf{k}, \mathbf{r})}$$

$$\bar{v}(\mathbf{r}, t) = \sum_{\mathbf{k}} v_{\mathbf{k}}(t) G(\mathbf{k}) e^{-i(\mathbf{k}, \mathbf{r})}$$

$$G^2(k) = \frac{1}{1 + \left(\frac{k}{k_c}\right)^{2n}}$$

$$\tilde{v}(\mathbf{r}, t) = v(\mathbf{r}, t) - \bar{v}(\mathbf{r}, t)$$

Butterworth low-pass filter[1]



[1] https://en.wikipedia.org/wiki/Butterworth_filter

Space Filtering & Energy Flux

$$\overline{\frac{\partial \vec{v}}{\partial t} + (\vec{v}, \nabla)\vec{v}} = \overline{-\frac{\nabla p}{\rho} - \alpha\vec{v} + \nu\Delta\vec{v} + \vec{f}} \qquad | \times \bar{v}_i, \ \int dS$$

$$\frac{\partial E}{\partial t} = -\Pi - 2\gamma E + \Gamma$$

$E = (2S)^{-1} \int \bar{v}^2 \, dS$ **!**

$\gamma(k) = \alpha + \nu\langle k \rangle^2$

$\Gamma(k) = S^{-1} \int \bar{f}_i \bar{v}_i \, dS$

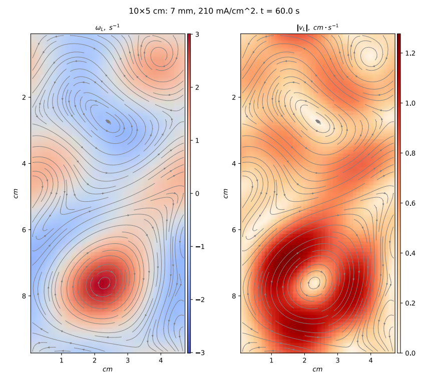
$\Pi(k) = -S^{-1} \int \tilde{v}_i \tilde{v}_j \frac{d\bar{v}_i}{dx_j} \, dS$

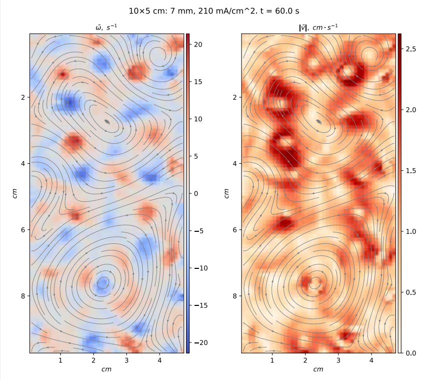
Direct cascade: $\Pi(k) > 0$, inverse cascade: $\Pi(k) < 0$.

In the inverse cascade ($k_{filter} \ll k_{force}$):

$$\frac{\partial E}{\partial t} \simeq 0, \ \Gamma \simeq 0 \quad \Rightarrow \quad \Pi + 2\gamma E \simeq 0, \quad \frac{\Pi}{E} \simeq -2\gamma$$

Space Filtering. Cell: 5×10 cm. $t - t_{on} = 60\,s$

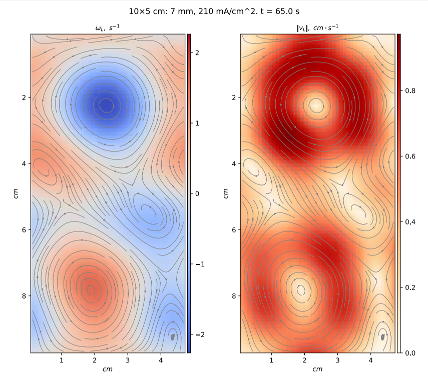


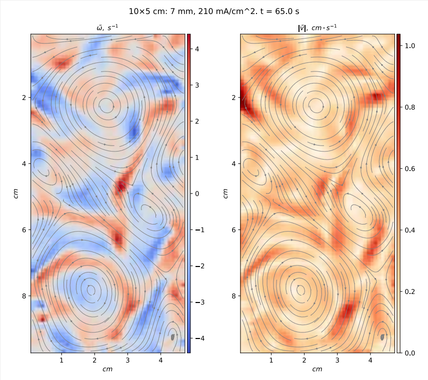


$\bar{\omega}(\mathbf{r}, t)$ $\bar{v}(\mathbf{r}, t)$ $\tilde{\omega}(\mathbf{r}, t)$ $\tilde{v}(\mathbf{r}, t)$

$$v(\mathbf{r}, t) = \bar{v}(\mathbf{r}, t) + \tilde{v}(\mathbf{r}, t) \qquad k_{filter} = 2.1\,cm^{-1}$$

Space Filtering. Cell: 5×10 cm. $t - t_{off} = 3\,s$



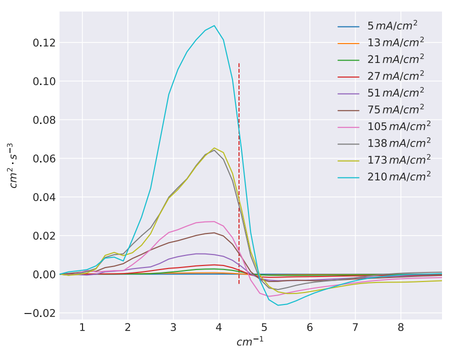


$\bar{\omega}(\mathbf{r}, t)$ $\bar{v}(\mathbf{r}, t)$ $\tilde{\omega}(\mathbf{r}, t)$ $\tilde{v}(\mathbf{r}, t)$

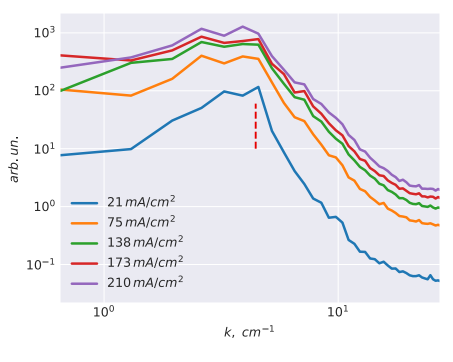
$$\Pi = -\tilde{v}_i \tilde{v}_j \frac{d\bar{v}_i}{dx_j}$$

Energy Flux. Cell: 10×10 cm

$-\Pi(k)$

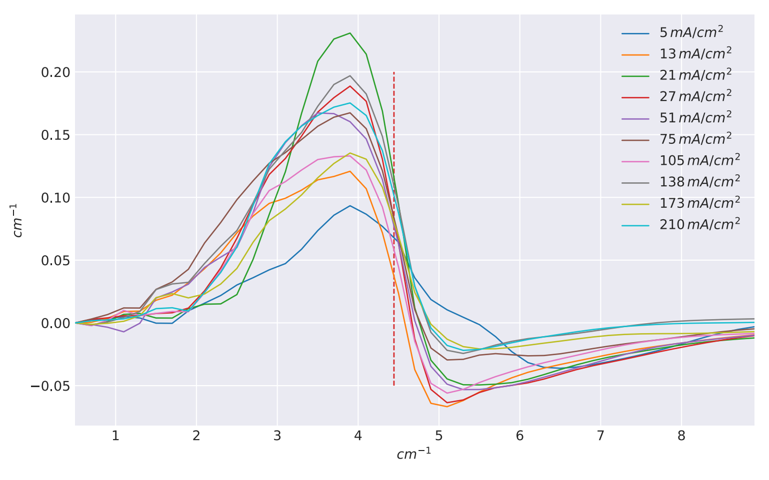


E_k



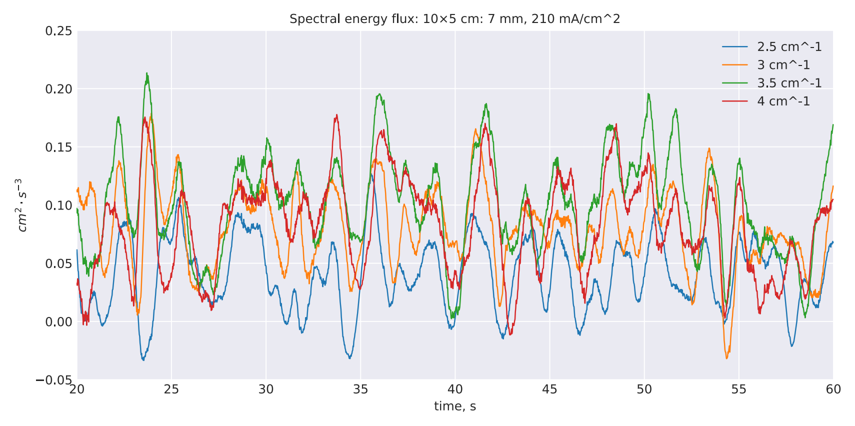
$$k_f = 4.4\,cm^{-1}$$

Energy Flux



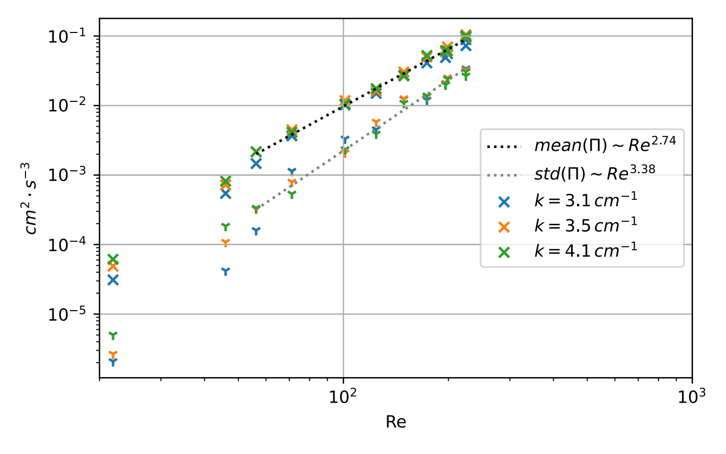
Normalized spectral flux $-\Pi(k)/E^{3/2}$

Energy Flux. Fluctuations



$$-\Pi(k,t)$$

Energy Flux

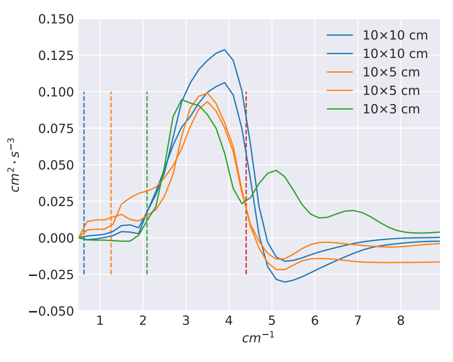


Mean spectral flux $\sim Re^{2.75\pm0.05}$

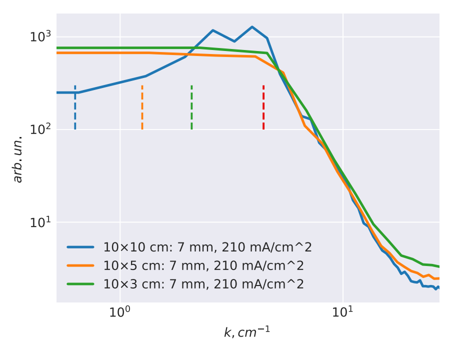
Standard deviation $\sim Re^{3.4\pm0.1}$

Energy Flux. $L_x = [3, 5, 10]\, cm$

$-\Pi(k)$

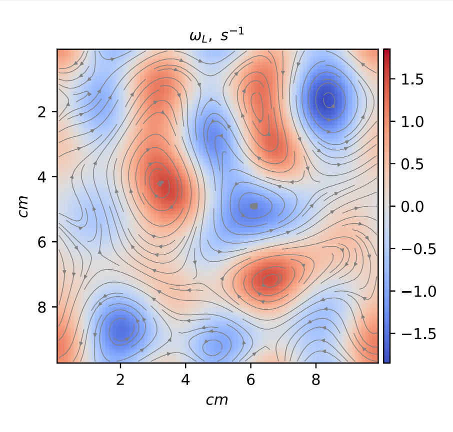


$E(k)$

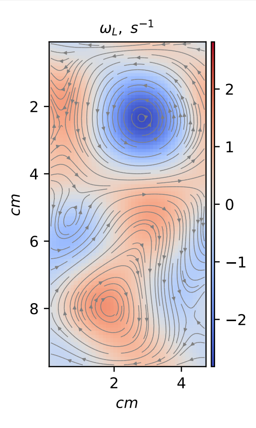


$$\Pi = 0.1\, cm^2 \cdot s^{-3}$$
$$E_c = 0.3\, cm^2 \cdot s^{-2}$$
$$\gamma = 0.077\, s^{-1}$$

$$L_c \sim \Pi^{1/2}\gamma^{-3/2} = 14.8\, cm > L_x$$
$$\Pi/E = 0.3\, s^{-1} > 2\gamma = 0.15\, s^{-1}$$



$L_x = 10\,cm$



$L_x = 5\,cm$

Vorticity of the large-scale flow.

Thank you!

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