# Бильярды с полужесткими стенками и
# и длинные стоячие нелинейные береговые волны

**С.Ю.Доброхотов, Д.С.Миненков, В.Е.Назайкинский**

Институт проблем механики им.А.Ю.Ишлинского РАН,
Центр интегрируемых систем,
Ярославский государственный университет им. П. Г. Демидова, Ярославль, Россия

Работа поддержана грантом РНФ 21-71-30011

**XXXI Научная сессия Совета РАН по нелинейной динамике**

**19-20 декабря 2022**

**Formulation of the problem**

$$\eta_t + \langle \nabla, (D(x)+\eta)\mathbf{u}\rangle = 0, \quad \mathbf{u}_t + \langle \mathbf{u}, \nabla\rangle \mathbf{u} + g\nabla\eta = 0, \quad x = (x_1, x_2)$$

$\eta = \eta(x,t)$ is the free elevation,

$\mathbf{u} = \mathbf{u}(x,t) = (\mathbf{u}_1(x,t), \mathbf{u}_2(x,t))$ is the velocity

$g$ is the free acceleration

the depth $D(x)$ is a smooth function in $\Omega^0 = \{x \in \mathbb{R}^2 \mid D(x) > 0\}$,

$\nabla D(x) \neq 0$ on $\Gamma^0 = \partial\Omega^0 = \{x \in \mathbb{R}^2 \mid D(x) = 0\}$,

The free boundary condition

$$(\eta(x,t) + D(x))|_{\Gamma(t)} = 0, \quad \Omega_t = \{x \mid \eta(x,t) + D(x) > 0\}$$

## Linearization and nonlinear effects near beach

**waves without collapse** $\eta \mapsto \varepsilon\eta$, $\mathbf{u} \mapsto \varepsilon\mathbf{u}$, $\varepsilon$ is small-

$$\eta_t + \langle \nabla, D(x)\mathbf{u} \rangle + \varepsilon \langle \nabla, \eta\mathbf{u} \rangle = 0, \quad \mathbf{u}_t + g\nabla\eta + \varepsilon\langle \mathbf{u}, \nabla \rangle \mathbf{u} = 0,$$

$$(\varepsilon\eta(x,t) + D(x))|_{\Gamma(t)} = 0.$$

## The modified simplified Carrier-Greenspane transformation: the algorithm

1. Construct the solution $(N(x,t), \mathbf{U}(x,t))$ of linearized system in $\Omega^0$

$$N_t + \langle \nabla, D(x)\mathbf{U}\rangle = 0, \qquad \mathbf{U}_t + g\nabla N = 0.$$

2. Construct the approximate solution $(\eta(x,t), \mathbf{u}(x,t))$ to the nonlinear system in the parametric form

$$x = y - N(y,t)\frac{\varrho(y)\nabla D(y)}{\|\nabla D(y)\|^2}, \qquad \eta = N(y,t), \qquad \mathbf{u} = \mathbf{U}(y,t),$$

here $\varrho(x)$ is cut-off function near the coast

*Dobrokhotov S. Yu., Minenkov D. S., Nazaikinskii V. E* Asymptotic Solutions of the Cauchy Problem for the Nonlinear Shallow Water Equations in a Basin with a Gently Sloping Beach // Russ. J. Math. Phys., 2022, 29, pp. 28–36.

**The wave equation**

$$\frac{\partial^2 N}{\partial t^2} = \langle \nabla, gD(x)\nabla N \rangle, \qquad x \in \Omega^0,$$

the solution in the form

$$N = Re(e^{i\omega t}\psi(x)), \qquad \Longrightarrow$$

the spectral problem in $\Omega^0$:

$$\mathcal{L}\psi = \lambda\psi, \quad \text{где} \quad \mathcal{L} = -\langle \nabla, gD(x)\nabla \rangle, \qquad \lambda = \omega^2.$$

$g = 1, \quad \omega \longrightarrow \omega/\mu \ll 1,$

$$\hat{\mathcal{H}}\psi \equiv -\mu^2 < \nabla, gD(x_1, x_2)\nabla\psi >= \omega^2\psi, \quad (x_1, x_2) \in \Omega.$$

## “Billiards with semirigid walls”

$\Lambda$ – Lagrangian manifold ("generalized Liouville torus")

$$\Lambda = \{H = \omega^2, F = c^2\}$$

$$H = D(x)(p_1^2 + p_2^2), \quad \{H, F\} = 0$$

The solution in the form the modified Maslov canonical operator+ simplification

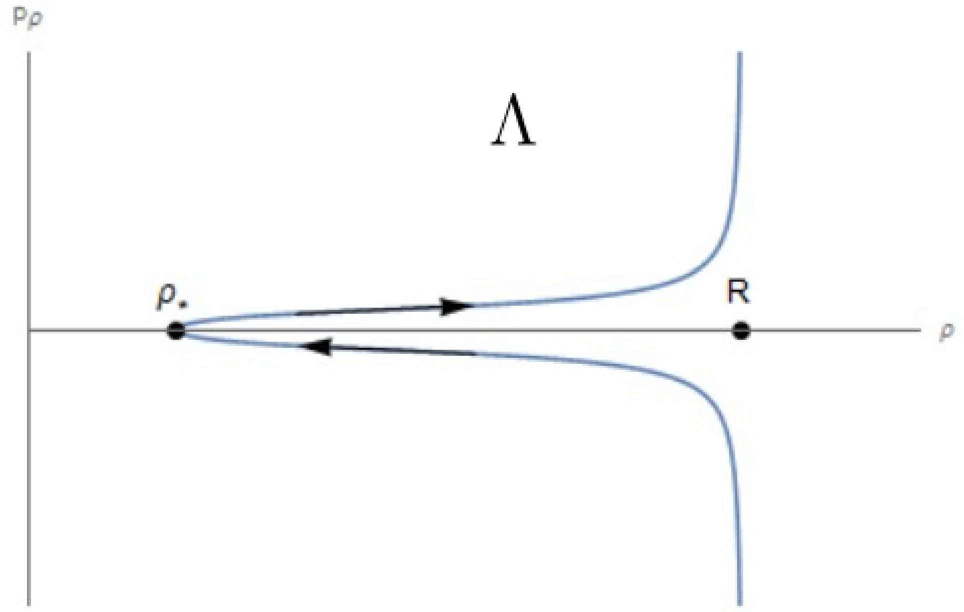
$$\psi(x) \approx K_\Lambda[A](x), \quad A \text{ – константа},$$

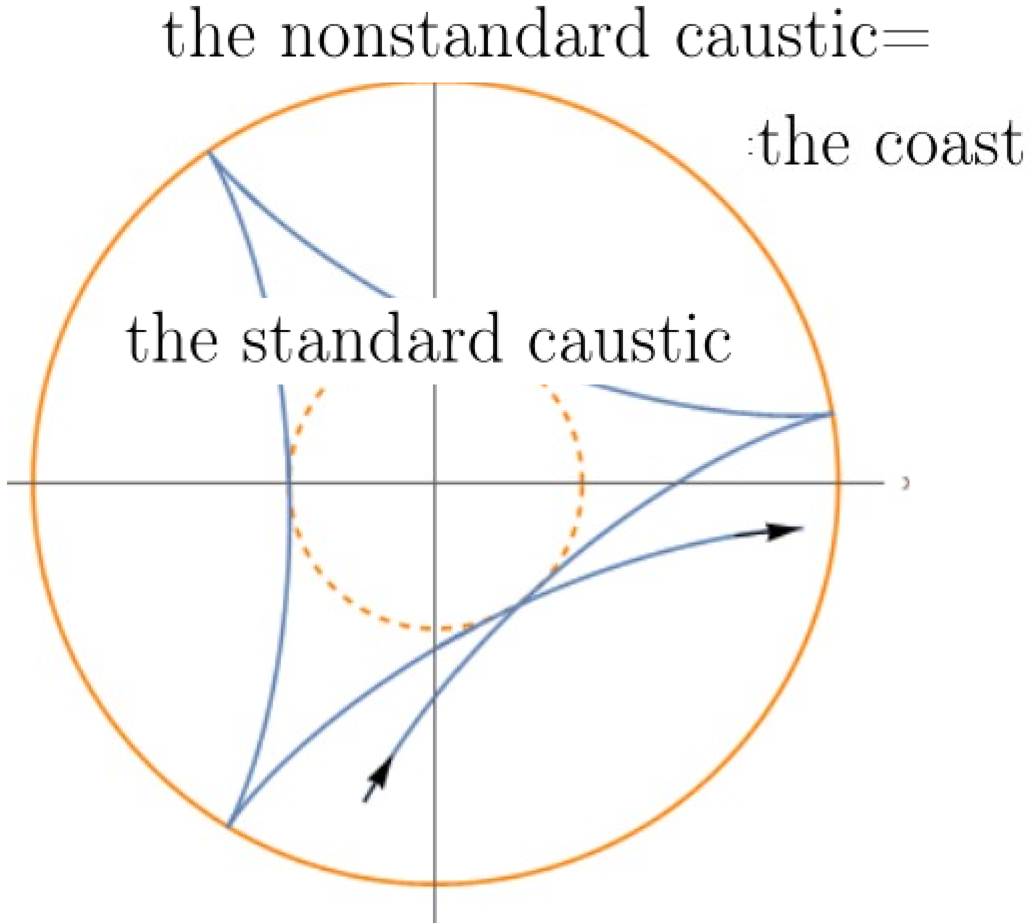
| *Аникин А.Ю., Доброхотов С.Ю., Назайкинский В.Е., Цветкова А.В.* Нестандартные лиувиллевы торы и каустики в асимптотиках в виде функций Эйри и Бесселя для двумерных стоячих береговых волн, Алгебра и анализ. 2021. Т. 33, Вып. 2. С. 5-–34.

# Example: the basin with a parabolic bottom

$$D(x_1, x_2) = D(\rho) = a(R^2 - \rho^2), \qquad a, R \text{ are parameters}$$

**The quantized frequencies** $\omega = \mu\sqrt{a(1+2n)(1+2n+2m)}$,
$m, n \sim \frac{1}{\mu} \gg 1.$



Near the coast

$$\psi(x) \approx \left. \frac{\mu^{-1/2}(-1)^n A e^{im\phi}\sqrt{2\pi}\,|\Psi(\rho)|^{1/2}}{(4a(R^2-\rho^2)(\rho^2(ac^2+\omega^2)-ac^2R^2))^{1/4}} \mathrm{J}_0\left(\frac{\Psi(\rho)}{\mu}\right)\right|_{\substack{\rho=\rho(x),\\ \phi=\phi(x)}},$$

here

$$\Psi(\rho) = \frac{1}{2\sqrt{a}}\left(\sqrt{ac^2}\arccos\left(\frac{2ac^2(R^2-\rho^2)-\rho^2\omega^2}{\rho^2\omega^2}\right)+\right.$$
$$\left.+\sqrt{ac^2+\omega^2}\arccos\left(\frac{-2ac^2(R^2-\rho^2)-R^2\omega^2+2\rho^2\omega^2}{R^2\omega^2}\right)-\pi\sqrt{ac^2}\right), \quad \rho \le R.$$

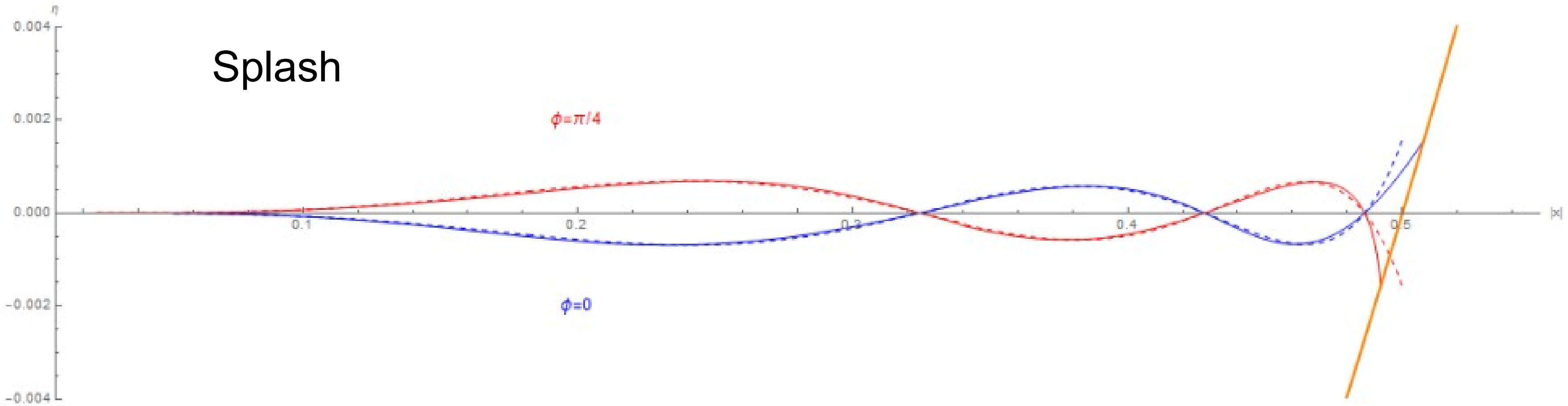
Near the standard caustic: $0 < \rho < R - \delta, \quad \delta > 0$ is small,

$$\psi(x) \approx \frac{\mu^{-1/6} A e^{im\phi} 2\sqrt{\pi}\,|\Phi(\rho)|^{1/4}}{(4a(R^2-\rho^2)(\rho^2(ac^2+\omega^2)-ac^2R^2))^{1/4}} \mathrm{Ai}\left(-\frac{\Phi(\rho)}{\mu^{2/3}}\right)\Bigg|_{\substack{\rho=\rho(x),\\ \phi=\phi(x)}},$$

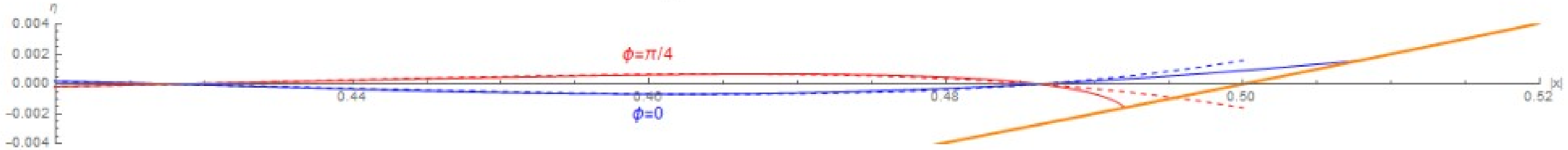
here

$$\Phi(\rho) = \begin{cases} \left(\frac{3}{2}\left(\frac{1}{2\sqrt{a}}\left(\pi\sqrt{ac^2+\omega^2} - \sqrt{ac^2}\arccos\left(\frac{2ac^2(R^2-\rho^2)-\rho^2\omega^2}{\rho^2\omega^2}\right) - \right.\right.\right. \\ \left.\left.\left. -\sqrt{ac^2+\omega^2}\arccos\left(\frac{-2ac^2(R^2-\rho^2)-R^2\omega^2+2\rho^2\omega^2}{R^2\omega^2}\right)\right)\right)\right)^{2/3}, & \rho \geq \rho_*, \\ \left(\frac{3}{2}\left(\frac{1}{2\sqrt{a}}\left(\sqrt{ac^2}\,\mathrm{arctanh}\left(\frac{2\sqrt{ac^2(\rho^2-R^2)(ac^2(-R^2+\rho^2)+\rho^2\omega^2}}{2ac^2(R^2-\rho^2)-\rho^2\omega^2}\right) + \right.\right.\right. \\ \left.\left.\left. \sqrt{ac^2+\omega^2}\,\mathrm{arctanh}\left(\frac{2\sqrt{(ac^2+\omega^2)(\rho^2-R^2)(ac^2(-R^2+\rho^2)+\rho^2\omega^2}}{2ac^2(-R^2+\rho^2)-R^2\omega^2+2\rho^2\omega^2}\right)\right)\right)\right)^{2/3}, & 0 < \rho < \rho_*. \end{cases}$$

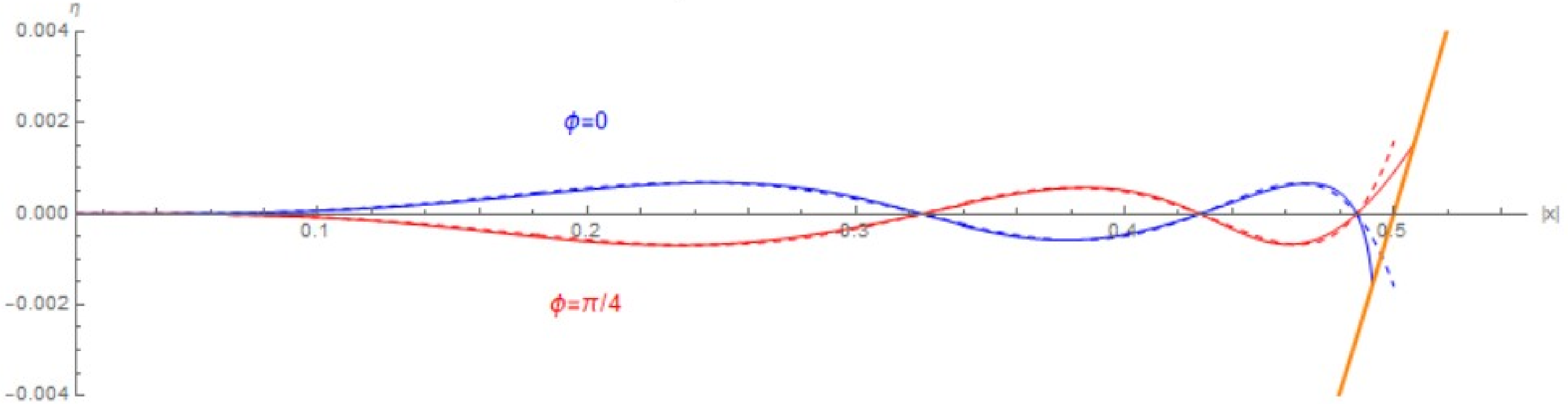
# Splash



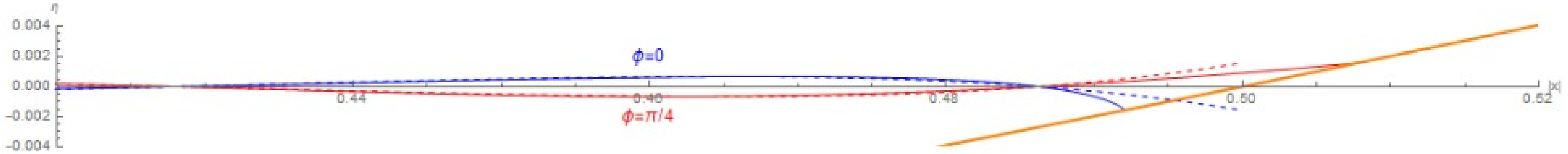

$t = 0$, масштаб осей $17:1$




$t = 0$, масштаб осей $1:1$




$t = 0.7$, масштаб осей $17:1$



## The waves trapped by an island

$$D(x_1, x_2) = \frac{x_1^2 + x_2^2}{f(\ln(\sqrt{x_1^2 + x_2^2})) + g(\arg(x_1 + ix_2))}.$$

$0 < u < 1$ , $f(u) = \frac{q(u)}{u(1-u)}$, $\quad q(u) > 0$, $\quad \Longrightarrow \quad D(x)$ defines the basin with the island $x_1^2 + x_2^2 = 1$ and the coast $x_1^2 + x_2^2 = e$.

Then

$$\begin{aligned} x_1 &= e^u \cos v \qquad & p_1 &= e^{-u}(\cos v p_u - \sin v p_v) \\ x_2 &= e^u \sin v \qquad & p_2 &= e^{-u}(\sin v p_u + \cos v p_v), \end{aligned}$$

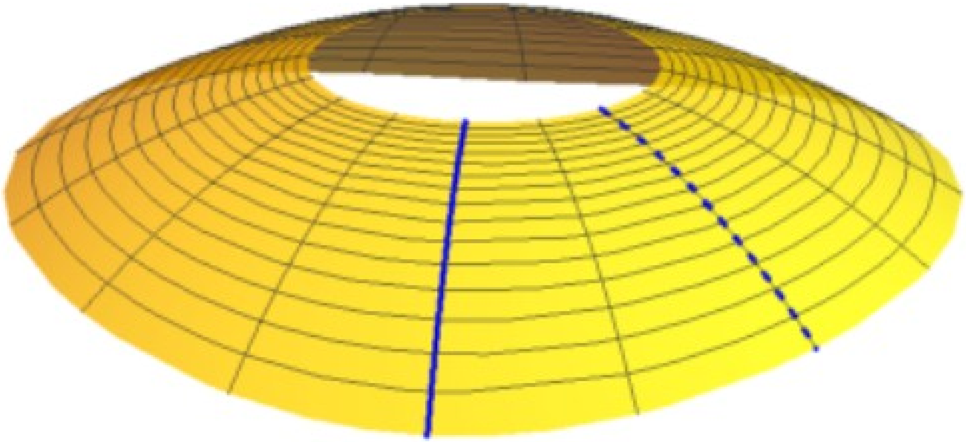
and

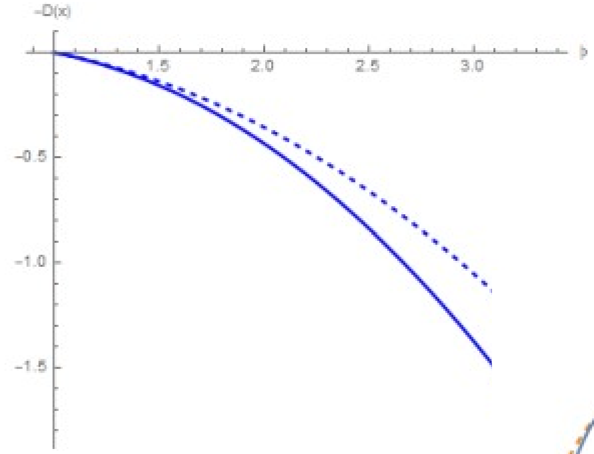
$$H = \frac{p_u^2 + p_v^2}{f(u) + g(v)}, \qquad F = \frac{g(v)p_u^2 - f(u)p_v^2}{f(u) + g(v)}.$$

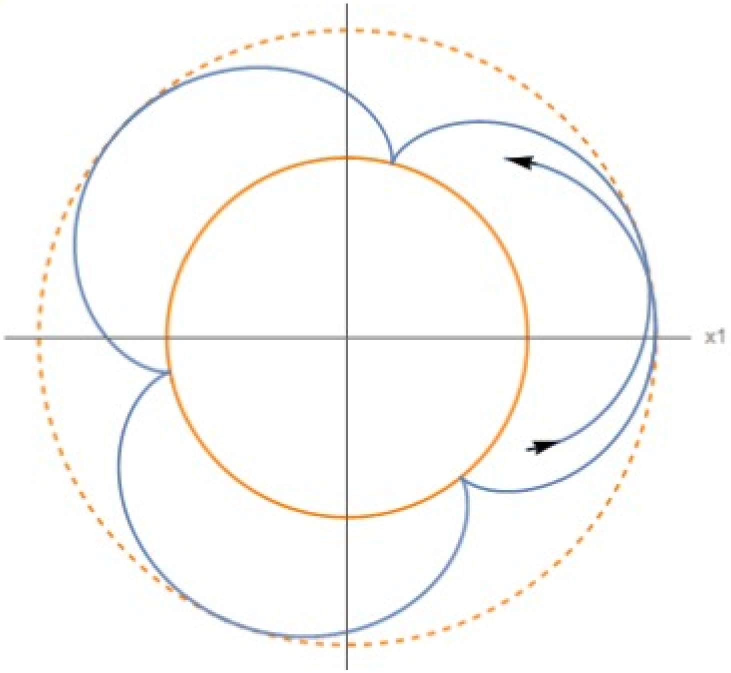
Матвеев В. С., *Асимптотические собственные функции оператора $\nabla D(x, y) \nabla$, отвечающие лиувиллевым метрикам, и волны на воде, захваченные донными неоднородностями*, Мат. заметки **64** (1998), №3, 414–422.
Тайманов И. А., *О первых интегралах геодезических потоков на двумерном торе*, Тр. Мат. ин-та РАН **295** (2016), 241–260.

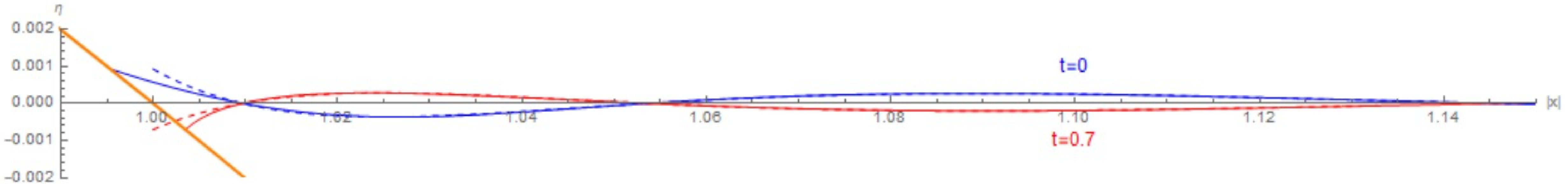
We take

$$f(u) = \frac{5}{u}, \quad g(v) = \sin^2 v + 2.$$

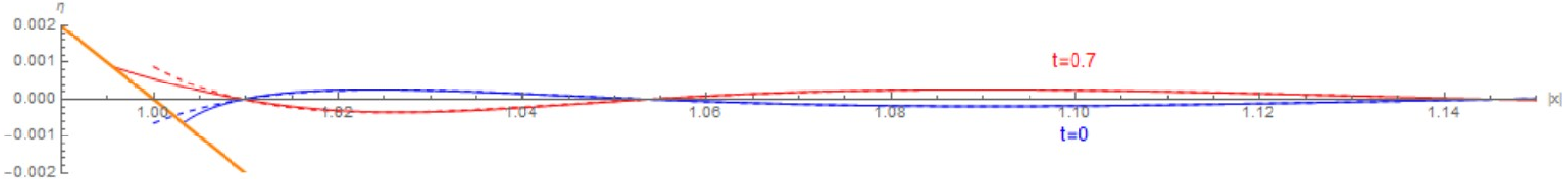





η
0.002
0.001
0.000
−0.001
−0.002
1.00
1.02
1.04
1.06
1.08
1.10
1.12
1.14
|x|
t=0
t=0.7

$$v = 0$$


η
0.002
0.001
0.000
−0.001
−0.002
1.00
1.02
1.04
1.06
1.08
1.10
1.12
1.14
|x|
t=0.7
t=0

$$v = \frac{\pi}{4}$$

**Спасибо за внимание!**

Будьте здоровы!