Frequency downshifting of decaying NLS solitons in an ocean covered by ice floes

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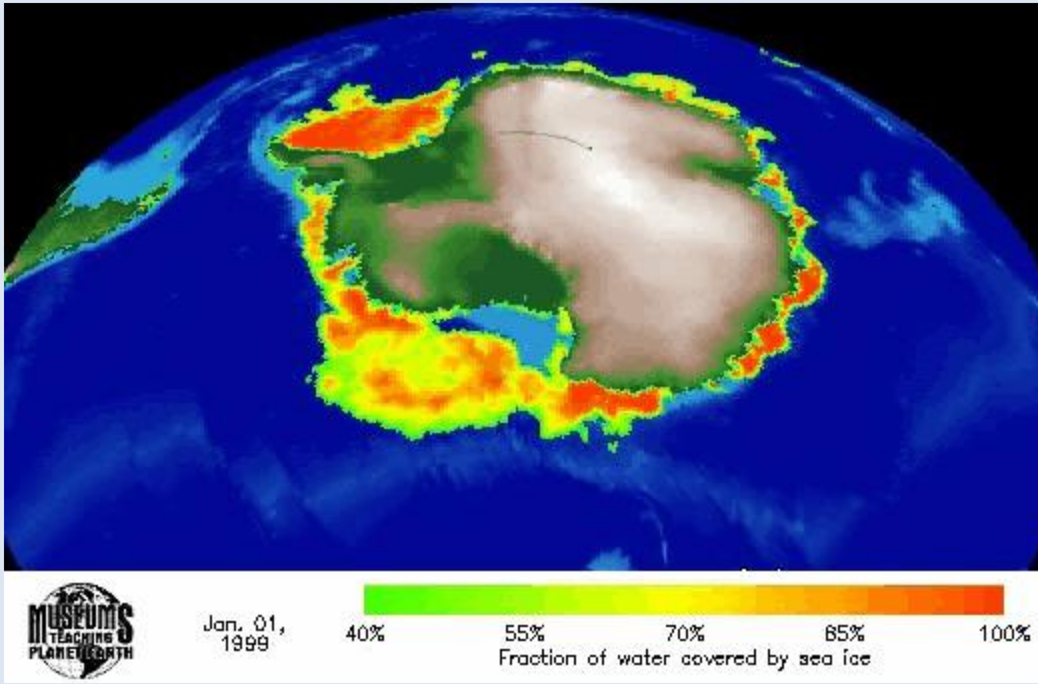
Oceans covered by ice

Sea ice covers about 12% of the world's oceans. Much of the world's sea ice is enclosed within the polar ice packs in the Earth's polar regions: the Arctic ice pack of the Arctic Ocean and the Antarctic ice pack of the Southern Ocean.

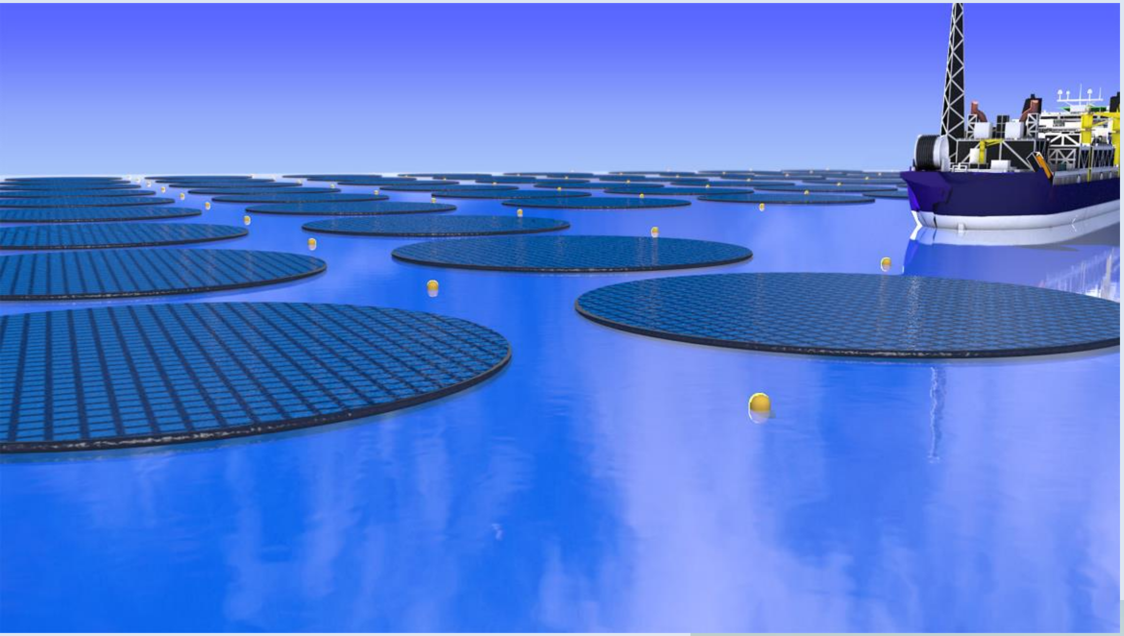
In winter seasons ice occupies significant ocean areas.

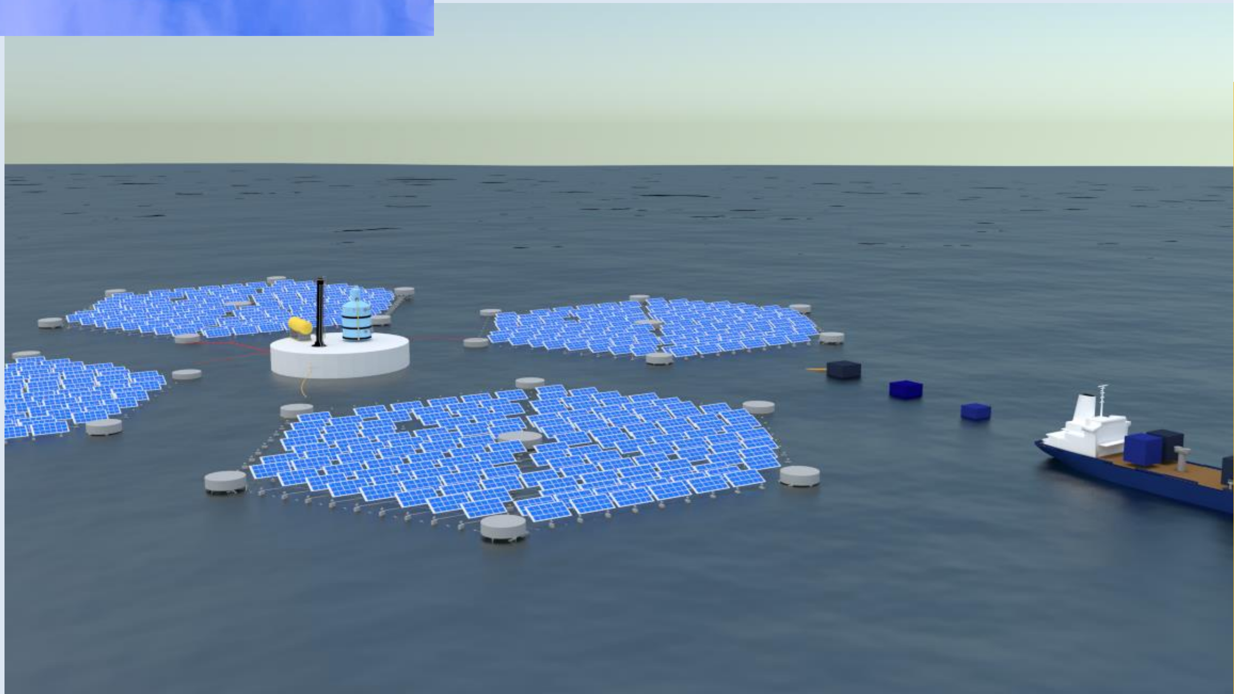


Ice floes

Photo from *https://xn----8sbiecm6bhdx8i.xn--p1ai/%D0%A1%D0%B5%D0%B2%D0%B5%D1%80%D0%BD%D1%8B%D0%B9%20%D0%9B%D0%B5%D0%B4%D0%BE%D0%B2%D0%B8%D1%82%D1%8B%D0%B9%20%D0%BE%D0%BA%D0%B5%D0%B0%D0%BD.html*

Floating solar power plants on the ocean





Surface wave decay in a viscous fluid

The well-know formula for the surface wave decay rate in a deep water is (*Landau & Lifshitz, 1987*):

$$\eta \sim e^{-2\nu k^2 t} \sim e^{-2\nu\omega^4 t/g^2} \sim e^{-4\nu\omega^5 x/g^3}; \qquad \omega = \sqrt{gk}.$$

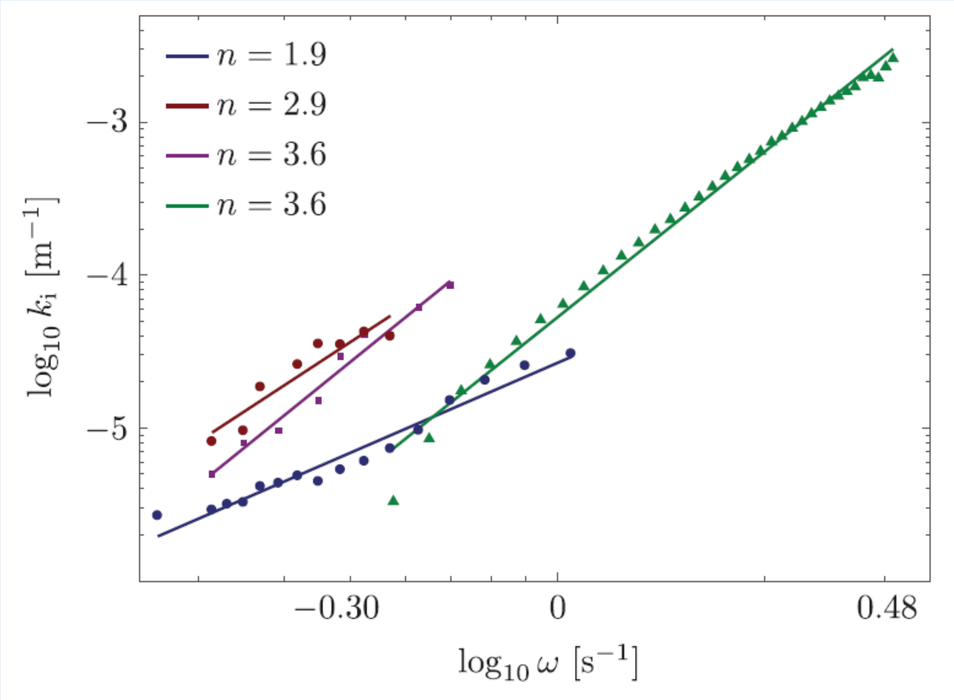
Therefore, the spatial decay rate is $\gamma = 4\nu\omega^5/g^3$.

Observations show that in the ice-covered ocean surface perturbation decay approximately exponentially in space $\eta \sim e^{-\gamma x}$.

In addition, frequency <u>downshifting</u> of decaying wavetrain is observed.

Surface wave decay in the ice-covered ocean

Data on decay rate measurements for surface waves in the ice-covered ocean are very uncertain: $\gamma \sim \omega^n$, where $1.9 < n < 3.6$ (Meylan et al., 2018).



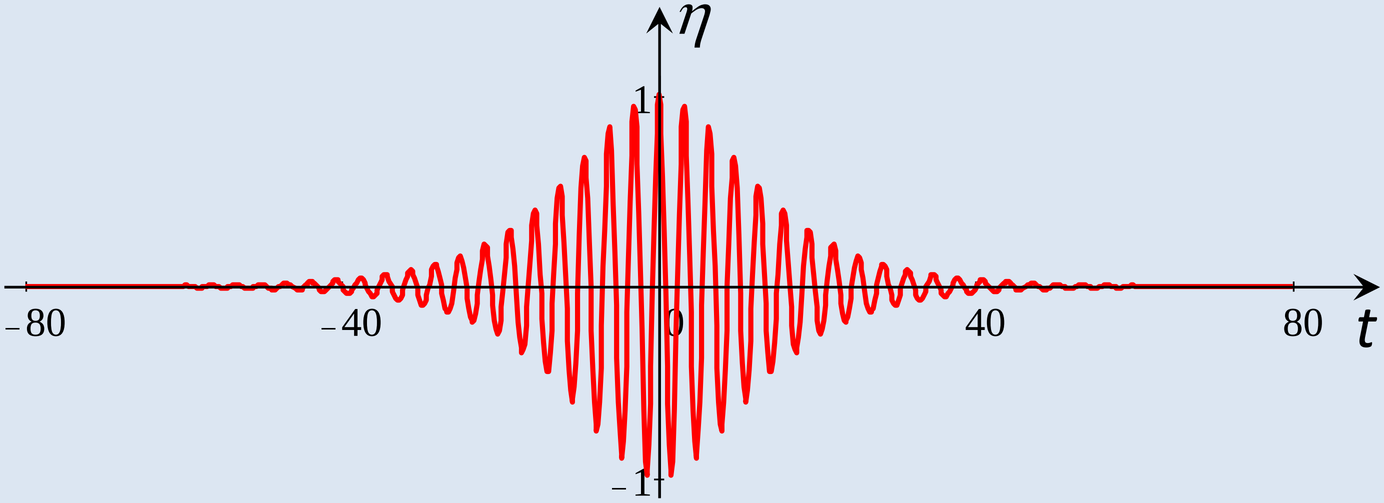
Further, we assume that $\gamma = \mu\omega^3$, albeit the theory is applicable to any power-type dependence with $n > 0$.

Frequency red-shifting (downshifting).

The linear theory

Frequency dependence of the decay rate leads to the frequency downshifting in the wave spectrum due to higher-frequency components decay faster than low-frequency components, $\gamma = \mu\omega^3$.

Example: $\eta(x,t) = A\,\mathrm{sech}\left[\frac{1}{T}\left(t - \frac{x}{V}\right)\right] e^{i(\omega_0 t - kx)}$



The Fourier-spectrum of the wavetrain is:

$$F_p(\omega) = \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \eta(x,t) e^{-i\omega t} dt \right| = \left| AT\sqrt{\frac{\pi}{2}} e^{-i[(\omega-\omega_0)x/V+kx]} \operatorname{sech}\left[(\omega-\omega_0)\frac{\pi T}{2} \right] \right|$$

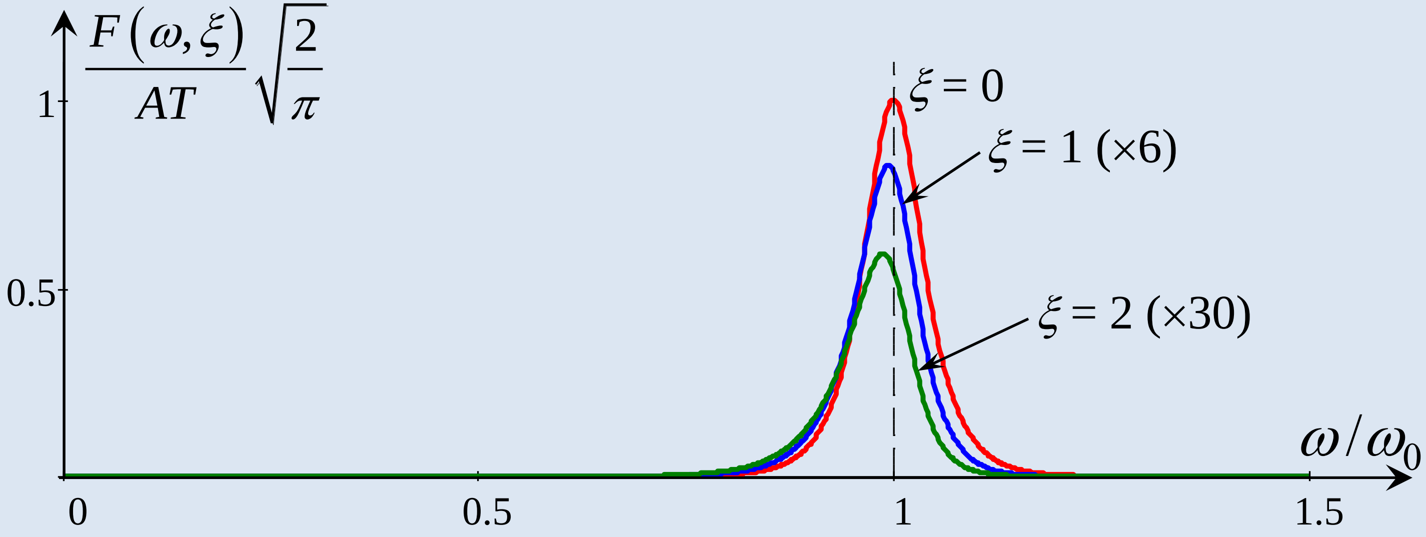
$$= AT\sqrt{\frac{\pi}{2}} \operatorname{sech}\left[(\omega-\omega_0)\frac{\pi T}{2} \right]$$

In the course of pulse propagation, its Fourier components changes

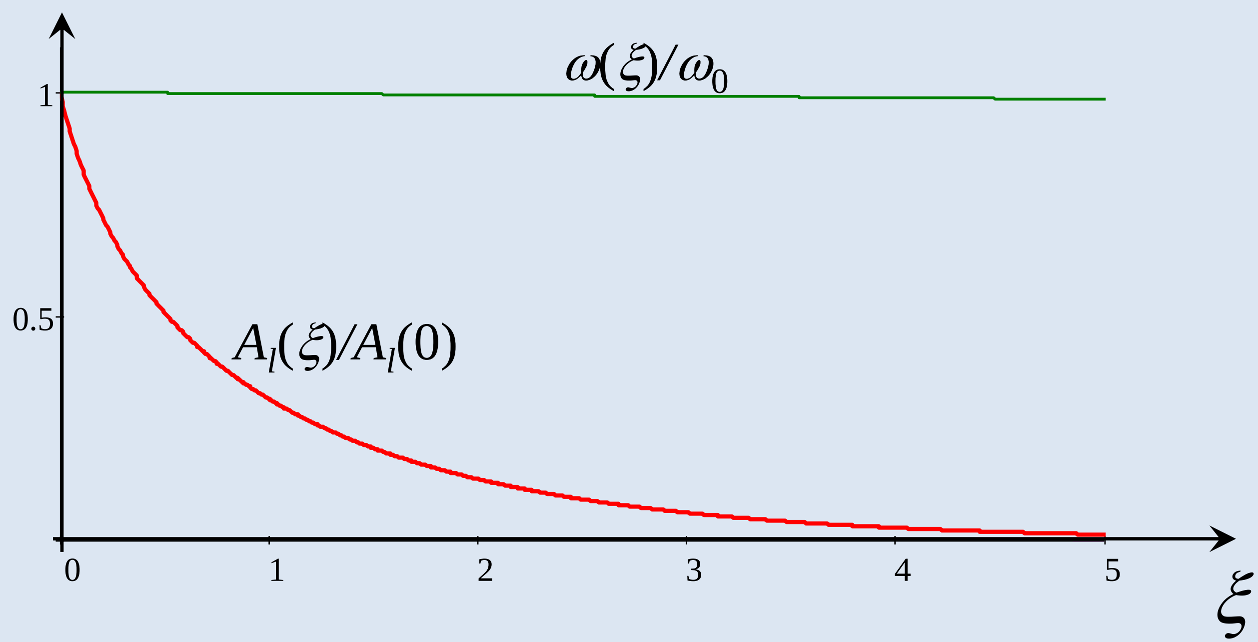
due to the non-uniform decay:

$$F(x,\omega) = F_p(\omega) e^{-\mu\omega^3 x} = AT\sqrt{\frac{\pi}{2}} \operatorname{sech}\left[\left(\tilde{\omega}-1\right)\frac{\pi\omega_0 T}{2} \right] e^{-\tilde{\omega}^3 \xi}$$

where $\omega = \omega/\omega_0, \quad \xi = \mu\,\omega_0{}^3 x.$



$$\xi = \mu\,\omega_0^{\,3} x$$



Frequency red-shifting (downshifting).

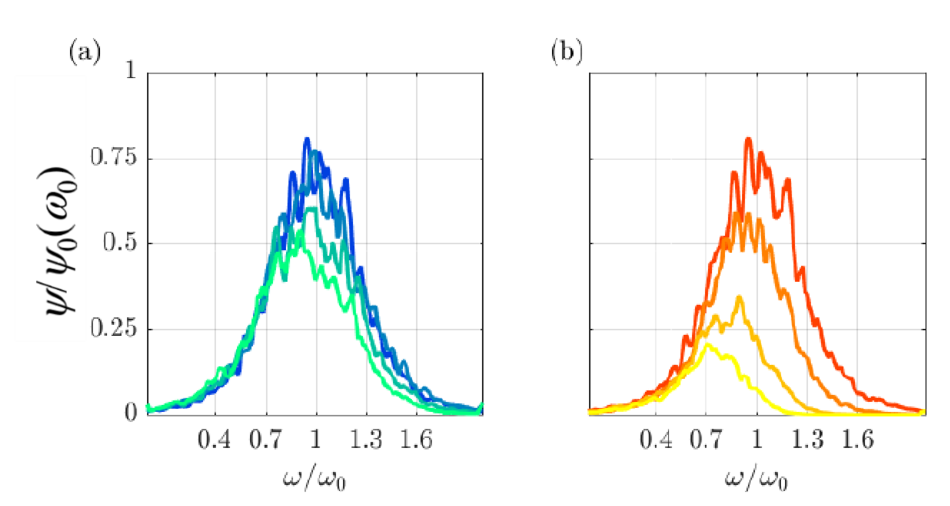
The NLS theory

Weakly nonlinear wavetrains can be described by the nonlinear Schrödinger (NLS) equation:

$$\boxed{i\left(\frac{\partial\psi}{\partial x}+\frac{1}{c_g}\frac{\partial\psi}{\partial t}\right)+\frac{\beta}{c_g^3}\frac{\partial^2\psi}{\partial t^2}+\frac{\alpha}{c_g}|\psi|^2\psi=-i\hat{R}(\omega)\psi}$$

Alberello & Parau (Phys. Fluids, 2022) considered downshifting numerically within the framework of the NLS equation with the Gaussian initial spectrum with $\hat{R}(\omega)=\gamma\omega^3$.

Downshifting was obtained for small (a) and big (b) dissipation as shown in the figure for $x = 0$, 5, 20, 50 km.

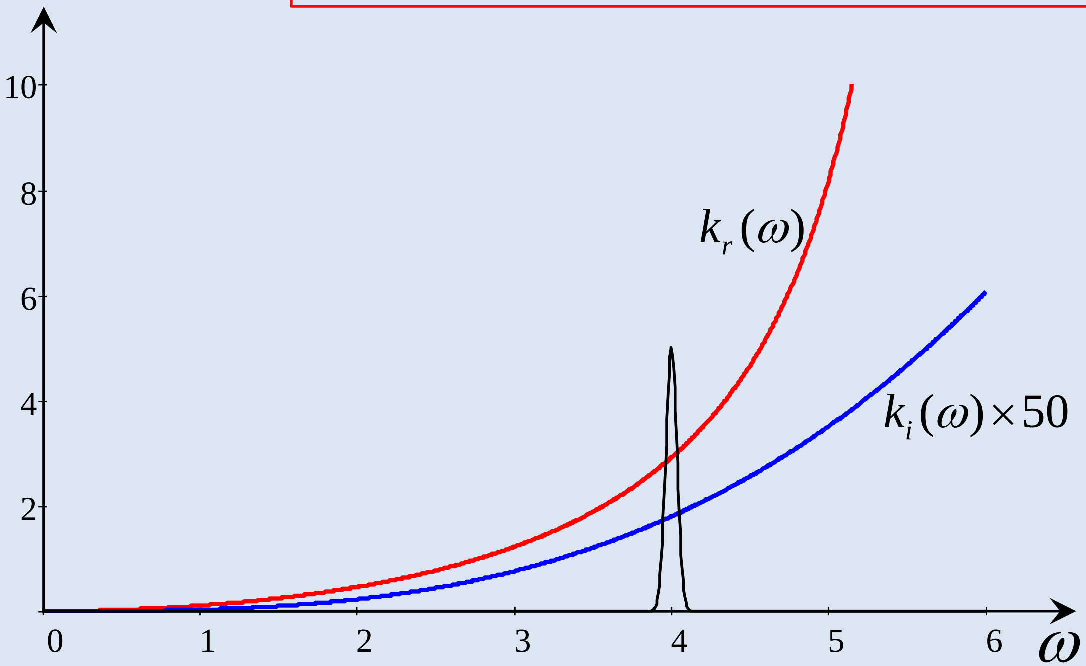


Frequency red-shifting (downshifting).

The NLS theory

We have reconsidered the problem within the NLS equation with the initial condition in the form of the narrow-band envelop NLS soliton :

$$\boxed{i\left(\frac{\partial\psi}{\partial x}+\frac{1}{c_g}\frac{\partial\psi}{\partial t}\right)+\frac{\beta}{c_g^3}\frac{\partial^2\psi}{\partial t^2}+\frac{\alpha}{c_g}|\psi|^2\psi=-i\hat{R}(\omega)\psi}$$



Real and imaginary parts of the dispersion relation.

$$\omega_r(k)=\sqrt{\frac{gk}{1+Mk}};\quad M=\frac{\rho_i}{\rho}h$$

$$k(\omega)=k_r(\omega)+ik_i(\omega);$$

$$k_r(\omega)=\frac{\omega^2}{g-M\omega^2};\quad \omega_c=\sqrt{\frac{g}{M}};$$

$$k_i(\omega)=-\mu\omega^3,\quad \mu=\frac{M\nu}{g^2}.$$

$$\rho_i=922.5\ \text{kg/m}^3,\quad \rho=1025\ \text{kg/m}^3,$$

$$h=0.3\ \text{m},\quad \nu=0.2\ \text{s}^{-1},$$

$$M=0.27\ \text{m},\quad \mu=5.6\cdot 10^{-4}\ \text{s}^3/\text{m}$$

Coefficients of the NLS equation has been derived in the paper

(Slunyaev, Stepanyants, Phys. Fluids, 2022):

$$\boxed{i\left(\frac{\partial\psi}{\partial x}+\frac{1}{c_g}\frac{\partial\psi}{\partial t}\right)+\frac{\beta}{c_g^3}\frac{\partial^2\psi}{\partial t^2}+\frac{\alpha}{c_g}|\psi|^2\psi=-i\hat{R}(\omega)\psi}$$

$$c_g=\frac{\left(g-M\omega_0^2\right)^2}{2g\omega_0},\quad \alpha=\frac{\omega_0^5}{4g^2}\frac{2g+M\omega_0^2}{g-M\omega_0^2},\quad \beta=\frac{1}{g+3M\omega_0^2}\left(\frac{g-M\omega_0^2}{2\omega_0}\right)^3.$$

The surface displacement is $\eta(x,t)=\mathrm{Re}\{\psi(x,t)\exp[i(\omega_0 t-k_0 x)]\}$;

$$k_i\approx-\mu[\omega_0^3+3\omega_0^2(\omega-\omega_0)].$$

$$\boxed{i\left(\frac{\partial\psi}{\partial x}+\frac{1}{c_g}\frac{\partial\psi}{\partial t}\right)+\frac{\beta}{c_g^3}\frac{\partial^2\psi}{\partial t^2}+\frac{\alpha}{c_g}|\psi|^2\psi=-i\mu\omega_0^2\left(\omega_0\psi-3i\frac{\partial\psi}{\partial t}\right).}$$

When $\mu = 0$, there is an exact solution to the NLS equation in the form of the envelop soliton with two independent parameters (say, A and σ):

$$\boxed{\psi(x,t) = A\,\mathrm{sech}\left[\frac{1}{T}\left(t - \frac{x}{V}\right)\right]e^{i(\sigma t - \kappa x)}}$$

$$T = \frac{1}{Ac_g}\sqrt{\frac{2\beta}{\alpha}}, \quad \sigma = \frac{c_g^2}{2\beta}\left(\frac{c_g}{V} - 1\right), \quad \kappa = \frac{c_g}{4\beta}\left(\frac{c_g^2}{V^2} - 1 - \frac{\alpha\beta}{2c_g^2}A^2\right).$$

When $\mu \neq 0$ but $\mu << 1$, this solution can be considered as an approximate with x-dependent parameters.

Dependence of parameters on x can be determined from the balance equations which follow from the NLS equation:

$$\frac{dN}{dx} = -2\mu\left(\omega_0^3 N - 6\omega_0^2 P\right); \quad \frac{dP}{dx} = -\mu\left(2\omega_0^3 P - 3\omega_0^2 D\right),$$

$$N = \int_{-\infty}^{+\infty} |\psi|^2 \, dt; \quad P = \frac{1}{2i} \int_{-\infty}^{+\infty} \psi^* \frac{\partial \psi}{\partial t} dt; \quad D = \int_{-\infty}^{+\infty} \left| \frac{\partial \psi}{\partial t} \right|^2 dt.$$

Substituting in the balance equation the soliton solution with $A(x)$ and $\sigma(x)$, we obtain:

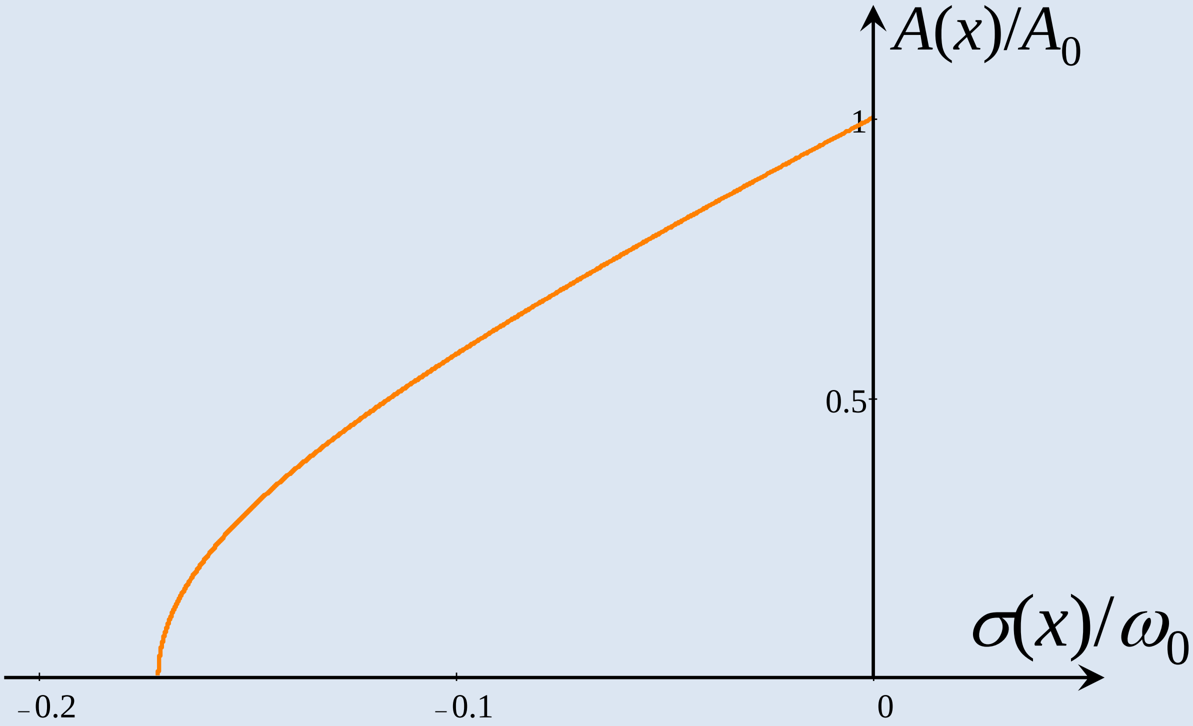
$$\frac{dA}{dx} = -2\mu\omega_0^3 A\left(1 + 3\frac{\sigma}{\omega_0}\right);$$

$$\frac{d\sigma}{dx} = -\mu\omega_0^2 c_g^2 \frac{\alpha}{\beta} A^2.$$

This system has the first integral:

$$A^2 = A_0^2 + \frac{4\beta\omega_0}{\alpha c_g^2}\sigma\left(1 + \frac{3}{2}\frac{\sigma}{\omega_0}\right).$$

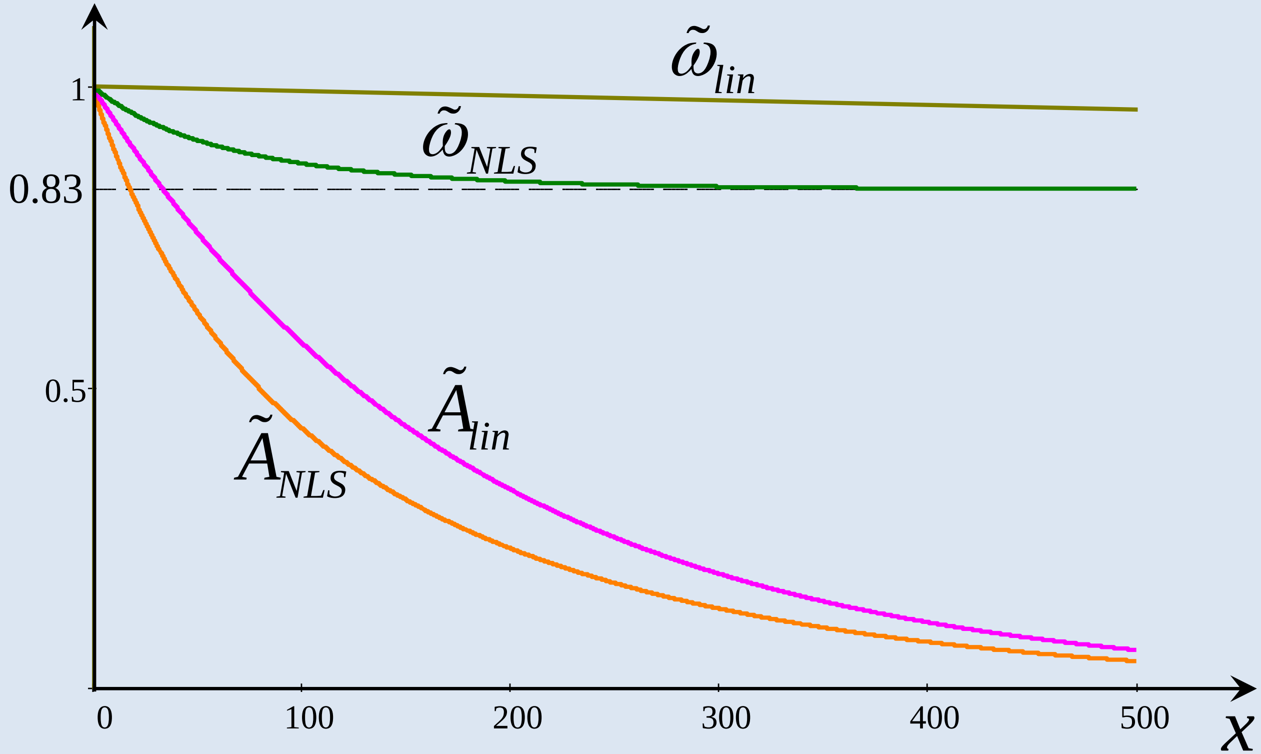
The dependence of soliton amplitude on the frequency shift is shown in the next figure.



Using the first integral, we can readily obtain the solution:

$$\boxed{\sigma(x) = -\frac{8}{27}\frac{\omega_0}{1+\sqrt{1-3\alpha c_g^2 A_0^2/2\beta\omega_0^2}}\left(1+\frac{3}{16}\frac{\alpha c_g^2 A_0^2}{\beta\omega_0^2}\right)\frac{1-e^{-12\mu\theta x}}{1-Re^{-12\mu\theta x}},}$$

$$\theta = \frac{\omega_0^3}{3}\sqrt{1-\frac{3}{2}\frac{\alpha c_g^2 A_0^2}{\beta\omega_0^2}}; \quad R = \frac{1-\sqrt{1-3\alpha c_g^2 A_0^2/2\beta\omega_0^2}}{1+\sqrt{1-3\alpha c_g^2 A_0^2/2\beta\omega_0^2}}.$$



Dependences of normalized amplitudes $\tilde{A} = A(x)/A(0)$ and frequencies $\tilde{\omega} = \omega(x)/\omega(0)$ within the linear and nonlinear theories.

Note, in the NLS case, the initial rate of amplitude decay two times greater than in the linear case.

The modified NLS theory

The theory presented above is valid only at the early stage of wavetrain evolution while the frequency deviation from initial value ω_0 is small, $|\sigma|/\omega_0 << 1$ because all coefficients of the NLS equation are frequency dependent.

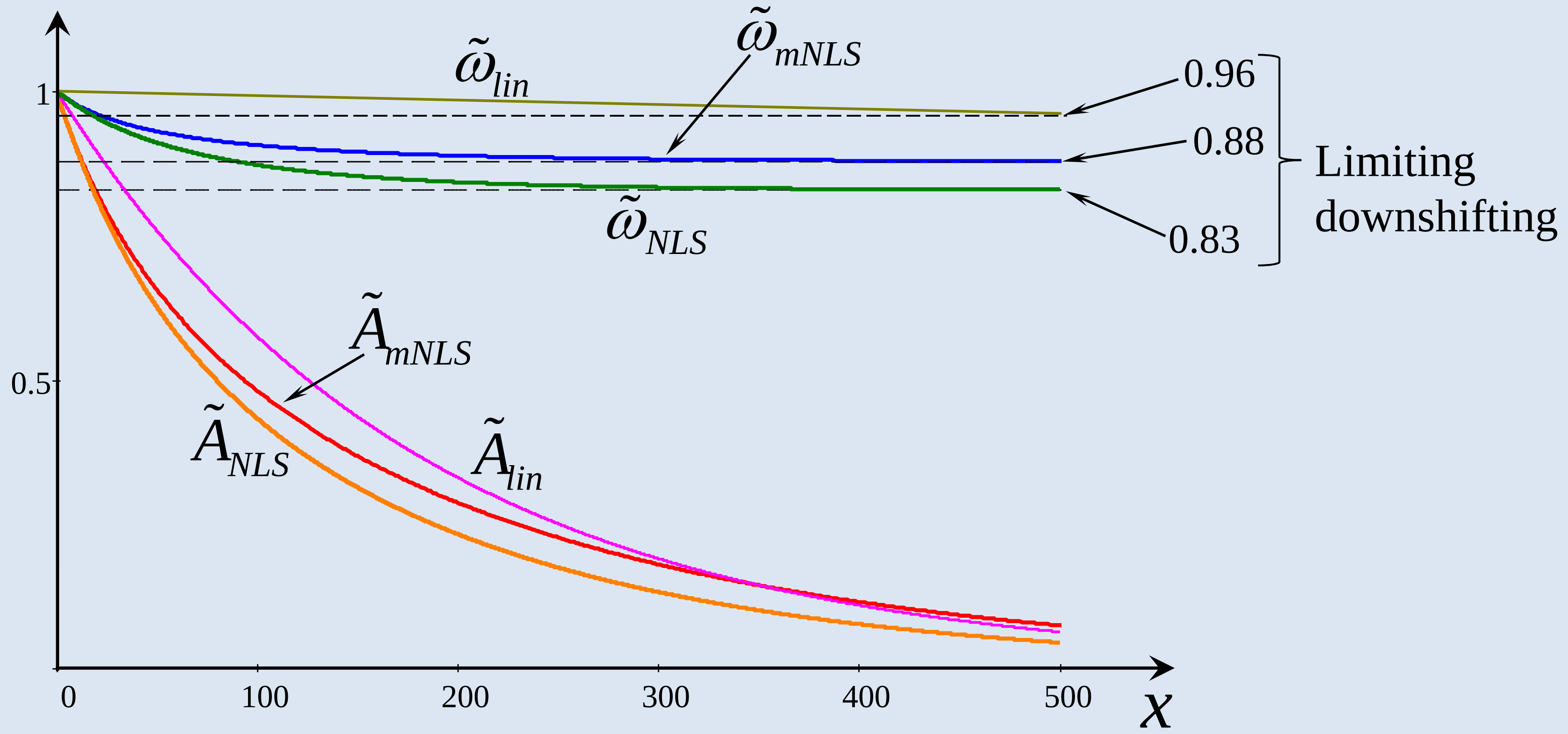
$$\boxed{i\left(\frac{\partial\psi}{\partial x}+\frac{1}{c_g}\frac{\partial\psi}{\partial t}\right)+\frac{\beta}{c_g^3}\frac{\partial^2\psi}{\partial t^2}+\frac{\alpha}{c_g}|\psi|^2\psi=-i\mu\omega^2\left(\omega\psi-3i\frac{\partial\psi}{\partial t}\right).}$$

$$c_g=\frac{\left(g-M\omega^2\right)^2}{2g\omega},\quad \alpha=\frac{\omega^5}{4g^2}\frac{2g+M\omega^2}{g-M\omega^2},\quad \beta=\frac{1}{g+3M\omega^2}\left(\frac{g-M\omega^2}{2\omega}\right)^3.$$

The asymptotic theory of soliton propagation remains applicable, but the set of equations becomes not solvable analytically.

$$\boxed{\begin{aligned} \frac{dA}{dx} &= -2\mu\left(\omega_0+\sigma\right)^3 A\left(1+3\frac{\sigma}{\omega_0+\sigma}\right); \\ \frac{d\sigma}{dx} &= -\mu\left(\omega_0+\sigma\right)^2 c_g^2(\omega_0+\sigma)\frac{\alpha(\omega_0+\sigma)}{\beta(\omega_0+\sigma)}A^2. \end{aligned}}$$

This system can be easily solved numerically.



Summary and conclusion

- In the course of wavetrain propagation in the ice-covered ocean, it experiences decay and downshifting.
- Within the linear theory, the downshifting occurs only within the width of the initial wavetrain spectrum due to the faster decay of high-frequency components. No energy flux along the spectrum occurs.
- Small-amplitude wavetrains with narrow-band spectra can be described by the NLS equation augmented by dissipative terms.
- Within the framework of weakly nonlinear theory, a wavetrain decays faster; the energy flux along the spectrum occurs which leads to greater frequency downshifting.