Solutions of the Euler equations and stationary structures in an inviscid fluid.

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It is well known that the equations

$$uu_x + vu_y + p_x = 0, \qquad uv_x + vv_y + p_y = 0, \qquad u_x + v_y = 0,$$

describe steady two-dimensional flows of inviscid fluid, where u, v are the components of velocity and p is the pressure. This system is reduced to one equation

$$\Delta\psi = \omega(\psi) \tag{1}$$

for the stream function ψ

Equation (1) is well investigated in the linear case. In addition, the elliptic Liouville equation

$$\Delta\psi = \exp(\psi)$$

connected by transformation

$$\psi = \log(-2\Delta(\log\tau))$$

with the Laplace equation $\Delta\tau = 0$. This is a direct consequence of the classical formula for the general solution of the hyperbolic Liouville equation

At the end of the nineteenth and beginning of the twentieth century, the Bäcklund and Tzitzeica [?] found transformations that allow generating solutions to the equations

$$u_{xy} = \sin(u), \qquad u_{xy} = \exp(u) - \exp(-2u).$$

Using the Bäcklund transformation, in [?] some vortex-type singular solutions of the elliptic Sine-Gordon equation

$$\Delta\psi = \sin(\psi) \tag{2}$$

were found. Multiparameter solution formula for the Tzitzeica equation

$$\Delta\psi = \exp(\psi) - \exp(-2\psi)$$

was presented in [?]. Note the works [?, ?], which used separation of variables to construct solutions of the equation (1) with other functions $\omega(\psi)$.

We will find some elementary solutions of equation (1), that is solutions which can be expressed in terms of algebraic operations, logarithms and exponentials. We begin with the Sine-Gordon and Sinh-Gordon equations

$$\Delta\psi = \sin(\psi), \tag{3}$$

$$\Delta\psi = \sinh(\psi). \tag{4}$$

Elementary solutions

Let us reduce these equations to a same form using complex and double numbers [?]. The field of complex numbers will be denoted by $\mathbb{C}$, and the algebra of double numbers by $\mathbb{D}$. Every complex and double number has the form $z = a + \delta b$, where $a, b \in \mathbb{R}$, $\delta \notin \mathbb{R}$. So if $\delta = i \in \mathbb{C}$, then δ^2 is equal to -1, and if $\delta \in \mathbb{D}$, then δ^2 is equal to 1. Multiplication of double numbers is given by:

$$(a_1 + \delta b_1)(a_2 + \delta b_2) = a_1 a_2 + b_1 b_2 + \delta(a_1 b_2 + a_2 b_1).$$

Thus, the equations (3), (4) can be written as

$$\Delta\psi = \frac{\exp(\delta\psi) - \exp(-\delta\psi)}{2\delta}.$$

Let $v = \exp(\delta\psi)$ be a new function, then the previous equation is of the form

$$v(v_{xx} + v_{yy}) - v_x^2 - v_y^2 - v^3/2 + v/2 = 0. \tag{5}$$

Suppose F and G are smooth functions on an open set $\Omega \subset \mathbb{R}^2$ and $H = F + \delta G$. Then we say that the function $\bar{H} = F - \delta G$ is conjugate to H. Next we look for solutions of (3), (4) in the form

$$v = \frac{\bar{H}^2}{H^2}. \tag{6}$$

Obviously, the functions v and ψ can be written as

$$v = \left(\frac{1 - \delta \frac{G}{F}}{1 + \delta \frac{G}{F}}\right)^2, \qquad \psi = \frac{2}{\delta} \log \left(\frac{1 - \delta \frac{G}{F}}{1 + \delta \frac{G}{F}}\right).$$

Therefore the function ψ is

$$\psi = 4 \tan^{-1} \frac{G}{F} \ , \tag{7}$$

when $\delta = i \in \mathbb{C}$; but if $\delta \in \mathbb{D}$, then

$$\psi = 4 \tanh^{-1} \frac{G}{F} \ .$$

We note the useful statement about the amount of fluid crossing a closed curve. Suppose the stream function ψ satisfies the Sine-Gordon equation and has the form (7), where F and G generate a vector field $V = (F, G)$ in the domain $\Omega \subset \mathbb{R}^2$. Let $a \in \Omega$ be an zero of V and c is a small circle around the zero. Then the line integral

$$ind(a) \equiv \frac{1}{2\pi} \oint_c d \tan^{-1}(G/F)$$

is integer called Poincaré's index of the point a [?]. Thus, the source strength

$$\oint_c d\psi,$$

is equal to $8\pi ind(a) \in 8\pi\mathbb{Z}$, i.e., the source strength is quantized in this case. We say that a line integral

$$q = \frac{1}{2\pi} \oint_c d\psi$$

is the topological charge of the point a.

Let us suppose that a simple closed curve γ is the boundary of Ω and the vector field V has several zeros $a_1, \dots a_n$ with topological charges $q_1, \dots, q_n$. It follows from Poincaré's theorem [?] that

$$\oint_\gamma d\psi = 8\pi \sum_{j=1}^{n} ind(a_j) = 2\pi \sum_{j=1}^{n} q_j.$$

Thus, the flux of fluid volume across the closed curve γ is

$$Q \equiv \oint_\gamma d\psi = 2\pi \sum_{j=1}^{n} q_j \in 8\pi\mathbb{Z},$$

i.e., this means that Q is also quantized. A useful way to think of singular solutions with topological charges is as point defects in the fluid.

We now look for solutions of the equations (3) and (4) by using the function H of the form

$$1 + \delta \exp(kx + ny + \eta),$$

with $k, n, \eta \in \mathbb{R}$. It is easy to see that v satisfies the equation (5), if $k^2 + n^2 = 1$. In the case of the Sine-Gordon equation, the function ψ is smooth; its graph is a two-dimensional kink.

The streamlines $\psi = const$ corresponding to this solution are obviously straight lines. For simplicity we assume that $k = 1, n = \eta = 0$. The corresponding stream function and velocity components are

$$\psi = 4\tan^{-1}(\exp(x)), \qquad v_x = 0, \quad v_y = -\frac{2}{\cosh^2(x/2)}.$$

Then the solution may be interpreted as a steady jet that is parallel to the y-axis. When $kn \neq 0$ we also have a jet flow. We remark that in the case of the Sinh-Gordon equation, the solution ψ is discontinuous.

Next, we call any linear combination of the functions $\exp(kx + ny + \eta)$, with $k, n, \eta \in \mathbb{C}$, a Hirota function. Consider a Hirota function

$$H = 1 + \delta(f_1 + f_2) + \delta^2 s_{12} f_1 f_2, \tag{8}$$

where $f_i = \exp(k_i x + n_i y + \eta_i)$, $i = 1, 2$. Substituting the function (6) into left side of (5), we obtain a rational function whose numerator is a polynomial P in k_i, n_i, s_{12}. Equating the coefficients of P to zero, we have a system NAS of nonlinear algebraic equations. The system has a non-trivial solution

$$s_{12} = \frac{n_1 n_2 + k_1 k_2 - 1}{n_1 n_2 + k_1 k_2 + 1}, \qquad n_i^2 + k_i^2 = 1 \quad i = 1, 2.$$

Hence solutions of the equations (3), (4) are given by

$$\psi_1 = 4 \tan^{-1}\left(\frac{f_1 + f_2}{1 - s_{12} f_1 f_2}\right), \qquad \psi_2 = 4 \tanh^{-1}\left(\frac{f_1 + f_2}{1 + s_{12} f_1 f_2}\right).$$

We further consider only the solution ψ_1 Suppose that $\eta_1 = \eta_2 = 0$, $n_i, k_i \in \mathbb{R}$. Streamlines are shown in Figure 1. It may be interpreted as an interaction of two jets.

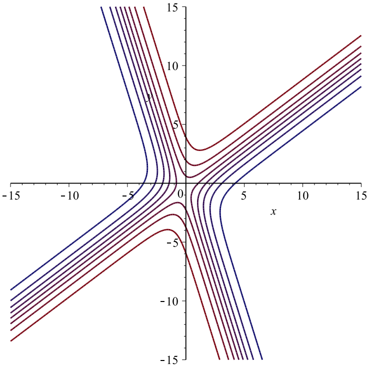

Рис.: two jets.

Now we set $\eta_1 = 0$, $\eta_2 = i\pi$, $n_i, k_i \in \mathbb{R}$. It gives a singularity of the velocity distribution at some point A, where $G = f_1 + f_2 = 0$ and $F = 1 - s_{12} f_1 f_2 = 0$. As we said above, this solution may be interpreted as a point source or sink in the rotational flow. Figure 2 shows the pattern of streamlines in the (x, y)-plane for the flow associated with a source or a sink.

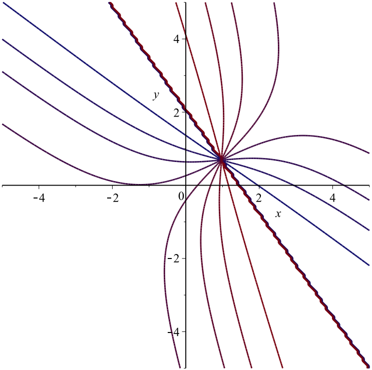

Рис.: source or sink.

Let us set $\eta_1 = \eta_2 = 0$, $n_1 = a + ib, n_2 = a - ib$ $(b \neq 0)$. Then the relation $f_1 + f_2 = 0$ gives countable number of straight and parallel streamlines. The perpendicular line defined by the equation $1 - s_{12} f_1 f_2 = 0$ is also a streamline. The points of intersection of the streamlines are sources or sinks. Each semi-infinite strip between the nearest parallel streamlines is filled with streamlines connecting a source and a sink.

Let us consider a Hirota function

$$\tau = 1 + \delta(f_1 + f_2 + f_3) + \delta^2(s_{12}f_1f_2 + s_{13}f_1f_3 + s_{23}f_2f_3) + \delta^3 s_{123}f_1f_2f_3. \tag{9}$$

Here, as above, the functions f_i have the form $\exp(k_ix + n_iy + \eta_i)$, with $k_i, n_i, \eta_i \in \mathbb{C}$, and s_{ij} is given by the following formula

$$s_{ij} = \frac{n_in_j + k_ik_j - 1}{n_in_j + k_ik_j + 1}, \qquad n_i^2 + k_i^2 = 1 \quad (i = 1, 2, 3). \tag{10}$$

Substituting the Hirota function (9) into the equation (5) and (6), we find

$$s_{123} = s_{12}s_{13}s_{23}.$$

To obtain flow patterns, one must again choose constants $n_i, \eta_i \in \mathbb{C}$ $(i = 1, 2, 3)$ and signs of $k_i = \pm\sqrt{1 - n_i^2}$. We have the simplest case when $\eta_i = 0$ and $n_i, k_i \in \mathbb{R}$ $(i = 1, 2, 3)$. If $n_1 = -0.7, n_2 = 0.4, n_3 = 0.1,$
$k_1 = \sqrt{1 - k_1^2}, k_2 = -\sqrt{1 - k_2^2}, k_3 = \sqrt{1 - k_3^2}$, then we get the flow pattern shown in Figure 3. There we see three jets and a vortex.

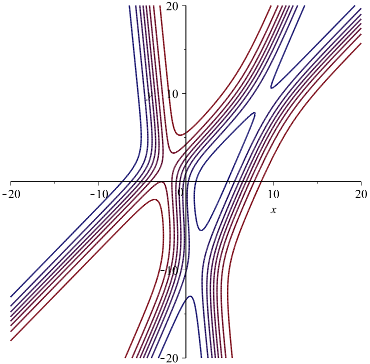

Рис.: three jets and a vortex

If we set $\eta_1 = i\pi, \eta_2 = \eta_3 = 0,\ n_1 = 0.1, n_2 = 0.4, n_3 = 0.3$ and $k_1 = \sqrt{1-k_1^2},\ k_2 = -\sqrt{1-k_2^2}, k_3 = \sqrt{1-k_3^2}$, then we have the flow pattern, including a sink, a source and jets (see Fig. 4). It is easy to construct other solutions by choosing, for example, n_1 and n_2 to be complex conjugate and n_3 to be real.

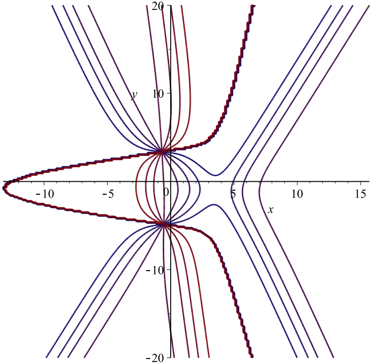

Рис.: a sink, a source and jets.

Formulas (8) and (9) can be written as

$$(1+\delta f_1)*(1+\delta f_2), \qquad (1+\delta f_1)*(1+\delta f_2)*(1+\delta f_3).$$

Here addition is defined in the usual way, and the operation $*$ is given by the formulas

$$f_i * f_j = s_{ij} f_i f_j, \qquad f_i * f_j * f_k = s_{ij} s_{ik} s_{jk} f_i f_j f_k,$$

where s_{ij}, s_{ik}, s_{jk} are calculated according to (10). For arbitrary n the Hirota function has the form of the product

$$H_n = (1+\delta f_1)*(1+\delta f_2)*\cdots*(1+\delta f_{n-1})*(1+\delta f_n).$$

In this case, the following relations must be fulfilled

$$f_{i_1} * \cdots * f_{i_m} = s_{i_1 \ldots i_m} f_{i_1} \cdots f_{i_m},$$

where $s_{i_1 \ldots i_m}$ is the product of all s_{jk} such that $j, k \in \{i_1, \ldots i_m\}$ and $j < k$.

Let us consider only some solutions to the Sine-Gordon equation for the case $n = 4$. The corresponding Hirota function is

$$H_4 = (1 + \delta f_1) * (1 + \delta f_2) * (1 + \delta f_3) * (1 + \delta f_4).$$

We set $n_1 = 0.5, n_2 = 0.4, n_3 = 0.3, n_4 = 0.2$, when $k_i < 0$ $(1 \leq i \leq 3)$, $k_4 > 0$. This gives a flow pattern with four jets and two vortices (see Fig. 5).

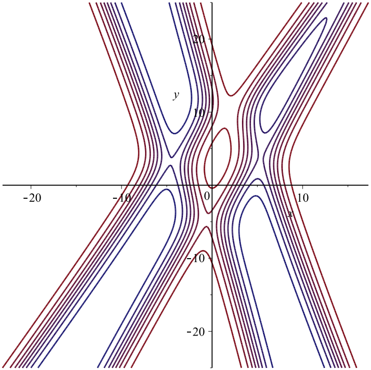

Рис.: four jets and two vortices.

the Tzitzéica equation

$$\Delta\psi = \exp(\psi) - \exp(-2\psi). \tag{11}$$

We have proposed in [?] the following representation

$$\psi = \log\left(1 - 2(\Delta \log H)\right)$$

of solutions of the Tzitzeica equation. The function ψ satisfies the equation (11) if H is the Hirota function

$$H = (1 + f_1) * (1 + f_2) * \cdots * (1 + f_n),$$

where $f_i = \exp(k_i x + n_i y + \eta_i)$, $n_i^2 + k_i^2 = 3$, $f_i * f_j = s_{ij} f_i f_j$,

$$s_{ij} = \frac{4k_i k_j n_i n_j + 4k_i^2 k_j^2 - 9n_i n_j - 6k_i^2 - 9k_i k_j - 6k_j^2 + 27}{4k_i k_j n_i n_j + 4k_i^2 k_j^2 + 9n_i n_j - 6k_i^2 + 9k_i k_j - 6k_j^2 + 27}.$$

Solutions expressed in elliptic functions

Let us now introduce a new function $w = \tan(\psi/4)$. Then the equation (3) is rewritten as

$$(1 + w^2)(w_{xx} + w_{yy}) - 2w(w_x^2 + w_y^2) + w^3 - w = 0. \qquad (12)$$

We look for solutions of the last equation in the form

$$w = \frac{s_0 + s_1 F + s_2 G}{p_0 + p_1 F + p_2 G}, \qquad (13)$$

with $s_i, p_i \in \mathbb{R}$ $(i = 0, 1, 2)$. Assume that $F(x), G(y)$ are functions satisfying ordinary differential equations

$$(F_x')^2 = a_4 F^4 + a_3 F^3 + a_2 F^2 + a_1 F + a_0, \qquad a_i \in \mathbb{R}, \qquad (14)$$

$$(G_y')^2 = b_4 G^4 + b_3 G^3 + b_2 G^2 + b_1 G + b_0, \qquad b_i \in \mathbb{R}. \qquad (15)$$

Substitute (13) into the left side of the equation (12) and express all derivatives using (14) and (15). As a result, we obtain a rational function of F and G. Equating the coefficients of the numerator to zero, we obtain a NAS system of nonlinear algebraic equations with respect to a_i, b_i $(0 \leq i \leq 4)$ and s_j, p_j $(0 \leq j \leq 2$). NAS system solutions are quite cumbersome and therefore we consider only two cases.

The first case is an analogue of the Steuerwald ansatz

$$w = \frac{G}{F}, \tag{16}$$

where the functions F, G satisfy the equations

$$(F_x')^2 = b_4 F^4 + (1 - b_2) F^2 + b_0, \quad (G_y')^2 = b_4 G^4 + b_2 G^2 + b_0, \quad b_0, b_2, b_4 \in$$

If $b_4 < 0$, then the functions are expressed in terms of the Jacobi function dn (the delta amplitude). Thus the functions F and G are periodic and vanish twice on the period. As a result, we have a partition of xy-plane into equal rectangles. There are sources (or sinks, respectively) at two opposite vertices of the rectangle; inside there is one saddle point, and streamlines connect sources and sinks.

The second solution has the form

$$w = \frac{F - G}{F + G}. \tag{17}$$

Moreover, the functions $F(x)$, $G(y)$ satisfy the equations

$$(F_x')^2 = b_4 F^4 - (1 + b_2) F^2 + b_0, \qquad (G_y')^2 = b_4 G^4 + b_2 G^2 + b_0.$$

Now we look for solutions to equation (12) in the form

$$w = \frac{s_0 + s_1 F + s_2 G + s_3 FG}{p_0 + p_1 F + p_2 G + p_3 FG} \ . \tag{18}$$

Assume that the functions F, G satisfy equations (14), (15). Substitute the w given by the formula (18) into the left side of (12) and express all derivatives using (14), (15). Then we have a rational function of F, G, and equating the numerator coefficients to zero, we obtain a nonlinear algebraic system with respect to a_i, b_i $(0 \leq i \leq 4)$ and s_j, p_j $(0 \leq j \leq 3\ )$.

Its solutions can be found using computer algebra systems. Here are some representations for the function w:

$$w = \frac{s_0 + s_1 F + s_0 s_3 G/s_1 + s_3 FG}{p_2 G}\,, \qquad w = \frac{s_1 F + s_3 FG}{p_2 G + p_3 FG}\,,$$

$$w = 4\frac{a_0 s_1 F + s_3 FG}{p_0(4a_0 + a_1 F)}\,, \qquad w = \frac{s_1 s_2/s_3 + s_1 F + s_2 G + s_3 FG}{p_0}\,.$$

Coefficients of ordinary differential equations for the functions F, G are cumbersome and we do not present them.

Back to the Tzitzéica equation (11) and introduce a new function

$$v = \exp(\psi).$$

Then the equation (11) is rewritten as follows

$$v(v_{xx} + v_{yy}) - v_x^2 - v_y^2 - v^3/2 + 1 = 0. \tag{19}$$

First, we look for solutions to this equation in the form

$$v = \frac{s_0 + s_1 F + s_2 G}{p_0 + p_1 F + p_2 G}, \tag{20}$$

where F and G satisfy the equations (14), (15).

Substitute v again into the right side (19) and repeating the reasoning above, we get the following representation

$$v = s_0 + F + G, \qquad s_0 \in \mathbb{R}.$$

The equations for the functions F and G have the form

$$(F_x')^2 = 2F^3 + \left(3s_0 + \frac{a_1 - b_1}{2s_0}\right) F^2 + a_1 F - s_0^3 + \frac{s_0(a_1 + b_1)}{2} + 1 - b_0,$$
$$(G_y')^2 = 2G^3 + \left(3s_0 + \frac{-a_1 + b_1}{2s_0}\right) G^2 + b_1 G + b_0.$$

with $s_0 \neq 0$. The solutions of the previous two equations are expressed in terms of the Weierstrass elliptic functions.

To get a specific solution, let's set $s_0 = 1$, $b_0 = -0.1$, $a_1 = b_1 = -0.2$. As the initial data, we choose the values $F(0) = G(0) = -0.2$. A two-dimensional contour graph of the function v is shown in Fig. 6. On one of the contour lines, the function v is equal to zero, and therefore the stream function ψ is not defined on it.

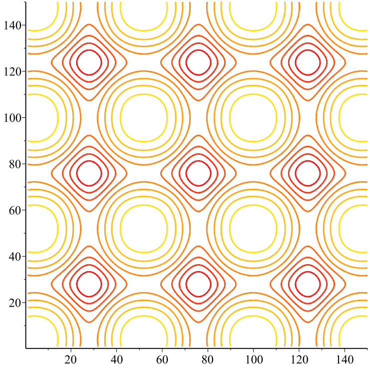

Рис.: contour plot of a solution of the equation (19).

We are now looking for a function v in the form

$$v = \frac{s_0 + s_1 F + s_2 G + s_3 FG}{p_0 + p_1 F + p_2 G + p_3 FG} \ . \tag{21}$$

By repeating the previous reasoning, we can obtain several different representations for v. The simplest of them is

$$v = \frac{p_0 + p_3 FG}{s_2 G} \ .$$

In this case, the functions F and G must satisfy the following equations

$$(F_x')^2 = 2p_3 F^3 - b_2 F^2 + b_3 p_0 F/p_3 + a_0, \tag{22}$$

$$(G_y')^2 = b_4 G^4 + b_3 F^3 + b_2 G^2 + 2p_0 G/s_2. \tag{23}$$

In conclusion, we give a few more representations for the function of the function v:

$$v = \frac{p_0 + p_1 F + p_3 FG}{s_1 F} \ ,$$

$$v = \frac{p_0 + p_1 F + p_0 s_3 G/s_1 + p_3 FG}{s_1 F + s_3 FG},$$

$$v = \frac{p_0 + p_1 F + p_2 G + p_3 FG}{p_0 s_3 G/p_1 + s_3 FG}.$$

We do not present the form of the corresponding equations (14), (15) due to their cumbersomeness.

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