QUANTISATION OF DYNAMICAL SYSTEMS ON FREE ALGEBRAS. QUANTISATION IDEALS

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Outline

NEWTON'S EQUATION $\ddot{q} = F(q)$

Classical Mechanics (Newton, Hamilton): $\mathfrak{A}_0 = \mathbb{C}[p, q]$

$$\partial_t q = p, \;\; \partial_t p = F(q), \quad H = \frac{p^2}{2} + U(q), \;\; F(q) = -U'(q)$$

$$\partial_t q = \{q, H\}, \; \partial_t p = \{p, H\}, \;\; \{q, p\} = 1, \;\; \partial_t a = \{H, a\}, \;\; a \in \mathfrak{A}_0.$$

Quantum Mechanics (Heisenberg): $\mathfrak{A}_\hbar = \mathbb{C}\langle \hat{p}, \hat{q}\rangle \diagup \langle \hat{p}\hat{q} - \hat{q}\hat{p} + i\hbar\rangle$

$$\partial_t \hat{q} = \hat{p}, \;\; \partial_t \hat{p} = F(\hat{q}), \quad H = \frac{\hat{p}^2}{2} + U(\hat{q}), \;\; F(\hat{q}) = -U'(\hat{q})$$

$$\partial_t \hat{q} = \frac{i}{\hbar}[H, \hat{q}], \; \partial_t \hat{p} = \frac{i}{\hbar}[H, \hat{p}], \quad [\hat{q}, \hat{p}] = i\hbar, \; \partial_t a = \frac{i}{\hbar}[H, a], \;\; a \in \mathfrak{A}_\hbar.$$

Pre-Quantum Mechanics or Free Associative Mechanics:
Free associative algebra $\mathfrak{A} = \mathbb{C}\langle p, q\rangle$ with a derivation

$$\partial_t : \mathfrak{A} \mapsto \mathfrak{A}, \qquad \partial_t(a\,b) = \partial_t(a)\,b + a\,\partial_t(b), \ \ \forall\, a, b \in \mathfrak{A}.$$

Nonabelian Newton's equations:

$$\partial_t q = p, \qquad \partial_t p = F(q) \qquad p, q \in \mathfrak{A}.$$

$H_0 = [q, p] \in \mathfrak{A}$ is a constant of motion

$$\partial_t([q, p]) = p^2 + qF(q) - F(q)q - p^2 = 0$$

of the nonabelian Newton's equations, but usual expression for the first integal of energy $H = \frac{1}{2}p^2 + U(q), \ F(q) = -U'(q)$ is not a constant of motion if $F''(q) \neq 0$.

Example: $F(q) = q^2,\ \ U(q) = -\frac{1}{3}q^3$. Then

- $H = \frac{1}{2}p^2 + U(q)$ is **not** a constant of motion

$$\begin{aligned}\partial_t(H) &= \tfrac{1}{2}(\partial_t(p)p + p\partial_t(p)) - \tfrac{1}{3}(\partial_t(q)q^2 + q\partial_t(q))q + q^2\partial_t(q)) \\ &= \tfrac{1}{6}(pq^2 - 2qpq + q^2p) \neq 0.\end{aligned}$$

- Nonabelian Newton's equation has higher symmetries. In our case ($F = q^2$) the next symmetry is:

$$\partial_\tau q = 3p^3 - 2pq^3 + qpq^2 + q^2pq - 2q^3p$$

$$\partial_\tau p = 2p^2q^2 - pqpq - pq^2p + 2qp^2q - qpqp + 2q^2p^2 - 2q^5$$

In the commutative case H is a first integral and $\partial_\tau = 6H\partial_t$.

Algebra $\mathfrak{A}$, as a $\mathbb{C}$–linear space has an additive basis of monomials

$$Mon(\mathfrak{A}) = \{p^{i_1}q^{j_1}p^{i_2}q^{j_2}\cdots p^{i_m}q^{j_m} \mid i_k, j_k \in \mathbb{N}\}, \qquad \mathbb{N} = \mathbb{Z}_{\geq 0},$$

and the number of monomials of a fixed degree n is growing exponentially, as 2^n.

In contrast, algebras $\mathfrak{A}_0 = \mathfrak{A}/\langle qp - pq\rangle$ and $\mathfrak{A}_\hbar = \mathfrak{A}/\langle pq - qp + i\hbar\rangle$ have additive bases of standard monomials

$$Mon(\mathfrak{A}_0) = Mon(\mathfrak{A}_\hbar) = \{p^i q^j \mid i, j \in \mathbb{N}\}$$

respectively, and the # of monomials of degree n is growing as $n + 1$.

Any element of the quotient algebra $\mathfrak{A}_0$ or $\mathfrak{A}_\hbar$ can be uniquely represented by a polynomial with standard monomials.

Fact: Any associative $\mathbb{C}$– algebra can be represented (is isomorphic to) a quotient of a free algebra $\mathfrak{A}$ over a two sided ideal $\mathfrak{J}$.

On my view the problem of *quantisation* of a dynamical system $\partial_t : \mathfrak{A} \mapsto \mathfrak{A}$ can be formulated as following:

Find such ideals $\mathfrak{J} \subset \mathfrak{A}$ that

- Q1. The quotient algebra $\mathfrak{A} / \mathfrak{J}$ has an additive basis of standard monomials. In other words, we know how to change the order of any two variables.
- Q2. $\partial_t(\mathfrak{J}) \subseteq \mathfrak{J} \quad \Leftrightarrow$ the evolutionary derivation ∂_t induces a derivation of the quotient algebra $\mathfrak{A} / \mathfrak{J}$.

Ideals $\mathfrak{J}$ satisfying conditions Q1,Q2 are called *quantisation ideals* and the corresponding quotient algebras $\mathfrak{A} / \mathfrak{J}$ *quantised algebras*.

In our example of the Newton equation on $\mathfrak{A}$ a natural candidate for $\mathfrak{J}$ which implies Q1 is

$$\mathfrak{J} = \langle J := pq - \omega qp + \alpha p + \beta q + \gamma \rangle, \ \ \omega, \alpha, \beta, \gamma \in \mathbb{C}.$$

Then $\mathfrak{J} = \{\sum a_i J\, b_i \mid a_i, b_i \in \mathfrak{A}\}$ and Q2 ($\partial_t(\mathfrak{J}) \subseteq \mathfrak{J}$) implies that

$$\partial_t(J) = F(q)q + p^2 - \omega p^2 - \omega q F(q) + \alpha F(q) + \beta p \in \mathfrak{J} \Rightarrow \omega = 1, \alpha = \beta = 0$$

and therefore $J = pq - qp + \gamma$, where $\gamma \in \mathbb{C}$.

If we add the reality arguments in the consideration we would conclude that γ is pure imaginary, i.e. $\gamma = i\hbar$ and the Heisenberg quantisation is a unique possibility for the free associative mechanics.

VOLTERRA AND BOGOYAVLENSKY SYSTEMS

Let us consider nonabelian integrable systems: the Volterra chain (i) and the Bogoyavlensky *N*–chains (ii)

$$\text{(i) } \partial_t u = u_1 u - u u_{-1}, \quad \text{(ii) } \partial_t u = \sum_{k=1}^{N} (u_k u - u u_{-k}). \qquad (1)$$

These are infinite systems of equations.
We use standard notations
$u = u_0 = u(n,t),\ u_k = u_k(n,t) = u(n+k,t),\ n,k \in \mathbb{Z}$.

In equations (1) functions u_k are elements of a free associative algebra $\mathfrak{A} = \mathbb{C}\langle \ldots u_{-1}, u, u_1, \ldots \rangle$ with an infinite number of variables u_k and a natural automorphism $S : \mathfrak{A} \mapsto \mathfrak{A}$, generated by the shift operator $S(u_k) = u_{k+1},\ k \in \mathbb{Z}$, and $\partial_t S = S \partial_t$.

We begin with consideration of two-sided ideals $\mathfrak{J}_\omega \subset \mathfrak{A}$ generated by an infinite set of polynomials of the form

$$\mathfrak{J}_\omega = \langle \{ u_q u_p - \omega_{p,q} u_p u_q \,|\, p,q \in \mathbb{Z},\ p > q,\ \omega_{p,q} \in \mathbb{C}^\times \} \rangle$$

$$u_p u_q = \omega_{q,p} u_q u_p, \qquad q > p, \quad \omega_{p,q} \neq 0.$$

Proposition

Volterra system $\partial_t u = u_1 u - u u_{-1}$ can be restricted to $\mathfrak{A}_{\mathfrak{I}_\omega}$ if and only if $\omega_{n+1,n} = \alpha,\ \omega_{n,m} = 1,\ n - m \geq 2$.

$$u_n u_{n+1} = \alpha u_{n+1} u_n, \quad u_n u_m = u_m u_n, \ \ |n - m| \geq 2.$$

The non-abelian Volterra system has a symmetry

$$\frac{du}{d\tau} = u u_{-1} u_{-2} + u u_{-1} u_{-1} + u u u_{-1} - u_1 u u - u_1 u_1 u - u_2 u_1 u. \qquad (2)$$

Proposition

Equation (2) can be restricted to $\mathfrak{A}_{\mathfrak{I}_\omega}$ only in the following cases:

$$\begin{array}{llll} \text{(a)} & u_n u_{n+1} = \alpha u_{n+1} u_n, & u_n u_m = u_m u_n, & \\ & & & n - m \geq 2 \\ \text{(b)} & u_n u_{n+1} = (-1)^n \alpha u_{n+1} u_n, & u_n u_m = -u_m u_n, & \end{array}$$

Periodic closures of the chains $u_{k+M} = u_k$ with period M result in nonabelian systems on $\mathfrak{A}^M = \mathbb{C}\langle u_1, \dots, u_M\rangle$.
Let $M = 3$. Then the Volterra system

$$u_{1,t} = u_2 u_1 - u_1 u_3,\ u_{2,t} = u_3 u_2 - u_2 u_1,\ u_{3,t} = u_1 u_3 - u_3 u_2$$

an obvious constant of motion $H = u_1 + u_2 + u_3$ and infinitely many commuting symmetries

$$\begin{aligned}
u_{1,\tau_1} &= u_1^2 u_3 + u_1 u_3 u_2 + u_1 u_3^2 - u_2 u_1^2 - u_2^2 u_1 - u_3 u_2 u_1,\\
u_{1,\tau_2} &= u_1^3 u_3 + u_1^2 u_3 u_2 + u_1^2 u_3^2 + u_1 u_2 u_1 u_3 + u_1 u_3 u_1 u_3 + u_1 u_3 u_2^2\\
&\quad + u_1 u_3 u_2 u_3 + u_1 u_3^2 u_2 + u_1 u_3^3 - u_2 u_1^3 - u_2 u_1 u_2 u_1 - u_2 u_1 u_3 u_1\\
&\quad - u_2^2 u_1^2 - u_2^3 u_1 - u_2 u_3 u_2 u_1 - u_3 u_2 u_1^2 - u_3 u_2^2 u_1 - u_3^2 u_2 u_1\\
\cdots &= \cdots
\end{aligned}$$

Periodic Volterra systems with period *M* may admit inhomogeneous commutation relations:

$$u_q u_p = \omega_{p,q} u_p u_q + \sum_{r=1}^{M} \sigma_{p,q}^{r} u_r + \eta_{p,q}, \qquad 1 \le q < p \le M, \;\; \omega_{p,q} \neq 0.$$

PROPOSITION

Nonabelian periodical Volterra chain with period M admits $\mathfrak{I}_M$–quantisation iff the following commutation relations

$$\begin{aligned}
M = 3: \quad & u_n u_{n+1} = \alpha u_{n+1} u_n + \beta(u + u_1 + u_2) + \eta, \;\; n \in \mathbb{Z}_3; \\
M = 4: \quad & u_1 u_2 = \alpha u_2 u_1 + \beta u_2 + \gamma u_1 - \beta\gamma, \\
& u_1 u_3 = u_3 u_1 - \beta u_2 + \beta u_4, \\
& u_4 u_1 = \alpha u_1 u_4 + \beta u_4 + \gamma u_1 - \beta\gamma, \\
& u_2 u_3 = \alpha u_3 u_2 + \beta u_2 + \gamma u_3 - \beta\gamma, \\
& u_2 u_4 = u_4 u_2 - \gamma u_3 + \gamma u_1, \\
& u_3 u_4 = \alpha u_4 u_3 + \beta u_4 + \gamma u_3 - \beta\gamma; \\
M \ge 5: \quad & u_{n+1} u_n = \alpha u_n u_{n+1}, \\
& u_n u_m = u_m u_n, \;\; |n - m| > 1, \; n, m \in \mathbb{Z}_M.
\end{aligned}$$

take place. The constants $\alpha, \beta, \gamma, \eta \in \mathbb{C}, \; \alpha \neq 0$ *are arbitrary.*

In the case $M = 3$ the ideal of quantisation has three parameters of quantisation

$$\mathfrak{J}_3 = \langle u_n u_{n+1} - \alpha u_{n+1} u_n - \beta(u + u_1 + u_2) - \eta \,|\, n \in \mathbb{Z}_3,\ \alpha \neq 0\rangle.$$

On $\mathfrak{A}_{\mathfrak{J}_3}$ the quantum Volterra system can be written as

$$u_{k,t} = \frac{1}{1-\alpha}[H, u_k], \qquad H = u_1 + u_2 + u_3.$$

The symmetry ∂_{τ_1} on $\mathfrak{A}_{\mathfrak{J}_3}$ is not independent:

$$u_{k,\tau_1} = \frac{1}{1+\alpha}\left(u_{k,t}H + Hu_{k,t} + \frac{2(\alpha-1)\beta}{\alpha+1}u_{k,t}\right).$$

The system admits a Casimir operator ($[\mathcal{C}, u_k] = 0,\ k \in \mathbb{Z}_3$):

$$\begin{aligned}\mathcal{C} \quad &= (\alpha^2 - 1)u_3 u_2 u_1 + (\alpha\beta + \beta)(u_2 u_1 + u_3 u_1 + u_3 u_2)\\ &+\alpha\beta u_1^2 + \alpha^{-1}\beta u_2^2 + \alpha\beta u_3^2\\ &+u_1(\alpha\eta + \beta^2 + \eta) + u_2(\alpha^{-1}\eta + \beta^2 + \eta) + u_3(\alpha\eta + \beta^2 + \eta).\end{aligned}$$

Proposition

Nonabelian N–chain $\partial_t u = \sum_{k=1}^{N}(u_k u - u u_{-k})$ *admits* $\mathfrak{J}_\omega = \langle \{u_q u_p - \omega_{p,q} u_p u_q \,|\, p, q \in \mathbb{Z},\ p > q,\ \omega_{p,q} \in \mathbb{C}^\times\}\rangle$ *quantisation only in the case*

$$\omega_{n+k,n} = \alpha, \quad \text{where } 1 \le k \le N,\ \alpha \ne 0, \text{ and } \quad \omega_{n,m} = 1, \quad \text{for } n - m > N.$$

$$u_n u_{n+k} = \alpha u_{n+k} u_n,\ \ 1 \le k \le N \qquad u_n u_m = u_m u_n,\ \ |n-m| > N.$$

Proposition

There exists a modification

$$u_t = u_2 u_1 u^2 + u_1 u u_{-1} u - u u_1 u u_{-1} - u^2 u_{-1} u_{-2}$$

of the nonabelian $N = 2$ *Bogoyavlensky chain. It admits* $\mathfrak{J}_\omega$*–quantisation only in the case*

$$\omega_{3n+1,3m} = \alpha, \quad \omega_{3n+2,3m} = \beta, \quad \omega_{3n+3,3m} = \alpha^{-1}\beta^{-1},$$

$\alpha, \beta \in \mathbb{C}^\times,\ n \ge m,\ n, m \in \mathbb{Z}.$

$$u_{3m} u_{3n+1} = \alpha u_{3n+1} u_{3m},\ \ u_{3m} u_{3n+2} = \beta u_{3n+2} u_{3m},\ \ u_{3m} u_{3n+3} = \alpha^{-1}\beta^{-1} u_{3n+3} u_{3m},\ \ n \ge m.$$

In algebra $\mathfrak{A} = \mathbb{K}\langle u, v\rangle$ we consider ideals

$$\mathfrak{J} = \langle vu - \alpha uv - \delta u^2 - \beta u - \gamma v - \eta\rangle$$

and systems of two quadratic homogeneous equations

$$\left\{ \begin{array}{l} u_t = \alpha_1 u^2 + \alpha_2 uv + \alpha_3 vu + \alpha_4 v^2, \\ v_t = \beta_1 v^2 + \beta_2 vu + \beta_3 uv + \beta_4 u^2 \end{array} \right. \tag{3}$$

possesing a hierarchy of symmetries.

Let us first consider equations (3) possesing a cubic symmetry

$$\left\{ \begin{array}{l} u_\tau = \gamma_1 u^3 + \gamma_2 u^2 v + \gamma_3 uvu + \gamma_4 vu^2 + \gamma_5 uv^2 + \gamma_6 vuv + \gamma_7 v^2 u + \gamma_8 v^3, \\ v_\tau = \delta_1 u^3 + \delta_2 u^2 v + \delta_3 uvu + \delta_4 vu^2 + \delta_5 uv^2 + \delta_6 vuv + \delta_7 v^2 u + \delta_8 v^3 \end{array} \right. \tag{4}$$

PROPOSITION

Any non-triangular system (3) possessing a non-zero cubic symmetry of the form (4) is equivalent to one of the following systems which admits a quantisation ideal $\mathfrak{J}$ generated by the comutation relation:

$$\mathrm{A}_1: \quad \begin{cases} u_t = u^2 - uv \\ v_t = v^2 + vu - uv \end{cases} \qquad uv = vu,$$

$$\mathrm{A}_2: \quad \begin{cases} u_t = uv \\ v_t = vu \end{cases} \qquad vu = \alpha uv, \qquad H = \alpha u - v,$$

$$\mathrm{A}_3: \quad \begin{cases} u_t = u^2 - uv \\ v_t = v^2 - vu \end{cases} \qquad vu = uv - \gamma u + \gamma v, \qquad H = uv - \gamma u$$

$$\mathrm{A}_4: \quad \begin{cases} u_t = -uv \\ v_t = v^2 + uv - vu \end{cases} \qquad vu = uv - \gamma v, \qquad H = uv + \gamma u$$

$$\mathrm{A}_5: \quad \begin{cases} u_t = uv - vu \\ v_t = u^2 + uv - vu \end{cases} \qquad vu = uv + \delta u^2 + \beta u + \eta, \qquad ?$$

$$\mathrm{A}_6: \quad \begin{cases} u_t = v^2 \\ v_t = u^2 \end{cases} \qquad vu = uv + \eta, \qquad H = v^3 - u^3,$$

where $\alpha, \beta, \gamma, \delta, \eta \in \mathbb{K}$ are arbitrary constants and $\alpha \neq 0$ and H is the Hamiltonian.

PROPOSITION

Any non-triangular system (3) possessing a symmetry of degree four, but not of a qubic one admit $\mathfrak{J}$ quantisation with the following commutation relations:

$$\mathrm{B}_1 \qquad \begin{cases} u_t = -uv \\ v_t = v^2 + vu \end{cases} \qquad uv = vu + \gamma v, \quad H = 2uv + u^2 + \gamma u$$

$$\mathrm{B}_2 \qquad \begin{cases} u_t = -vu \\ v_t = v^2 + vu \end{cases} \qquad uv = vu + \gamma v, \quad H = 2uv + u^2 + \gamma u + 2\gamma v$$

$$\mathrm{B}_3 \qquad \begin{cases} u_t = u^2 - 2vu \\ v_t = v^2 - 2vu \end{cases} \qquad vu = uv + \eta, \quad H = u^2 v - uv^2,$$

$$\mathrm{B}_4 \qquad \begin{cases} u_t = u^2 - uv - 2vu \\ v_t = v^2 - 2uv - vu \end{cases} \qquad vu = uv, \quad ?$$

$$\mathrm{B}_5 \qquad \begin{cases} u_t = u^2 - 2uv \\ v_t = v^2 + 4vu \end{cases} \qquad vu = uv, \quad ?$$

where $\gamma, \eta \in \mathbb{K}$ are arbitrary constants.

Heisenberg equations:

$$\kappa u_t = [H, u], \qquad \kappa v_t = [H, v].$$

$$B_1 : \ \kappa = -2\gamma, \quad B_2 : \ \kappa = -2\gamma, \quad B_3 : \ \kappa = \eta.$$