# Unstable modes near NLS Akhmediev breather

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The author was supported by RSF grant No 18-11-00316

XXIX Scientific session of the Nonlinear Dynamics Council of RAS, December 14, 2020,

# Rogue waves

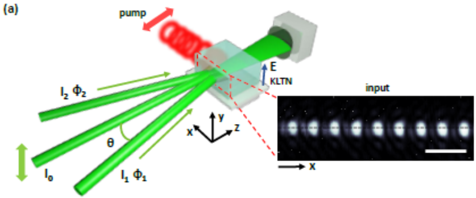
Rogue waves (freak waves, anomalous waves) in the ocean the great short-living waves appearing from almost nowhere.

Figure: Akademik Ioffe ship, Drake Straight

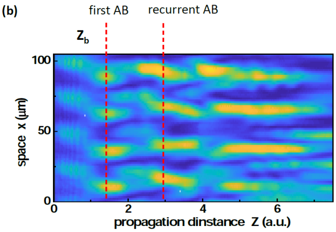
# Rogue waves

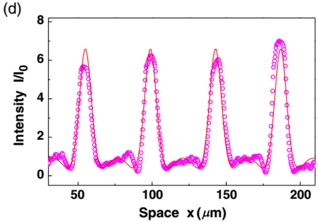
The RW recurrence in the periodic setting has been recently observed in experiments in water waves [Onorato et al ’13], in fiber optics [Trillo et al ’18], and in a photorefractive crystal:

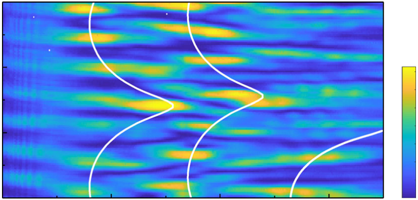
Pierangeli D., Flammini M., Zhang L., Marcucci G., Agranat A.J., Grinevich P.G., Santini P.M., Conti C., DelRe E. “Observation of Fermi-Pasta-Ulam-Tsingou recurrence and its exact dynamics”, Physical Review X, 2018, v. 8, issue 4, p. 041017 (9 pages); doi:10.1103/PhysRevX.8.041017;

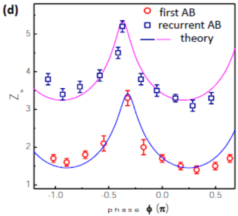



The symmetric 3-wave interferometric scheme used to generate the background wave with a single-mode perturbation propagating in a pumped photorefractive KLTN (potassium-lithium-tantalate-niobate) crystal.
Since NLS $i\psi_z + \psi_{xx} + 2|\psi|^2\psi = 0$ is supposed to describe the above physics only at the leading order, one expects that the exact NLS RW recurrence be replaced by a "Fermi-Pasta-Ulam" - type recurrence, before thermalization destroys the pattern.


(b)
first AB
recurrent AB
$Z_b$
100
50
0
space x (μm)
0
2
4
6
propagation dinstance Z (a.u.)


(d)
8
6
4
2
0
Intensity $I/I_0$
50
100
150
200
Space x (μm)




(d)
first AB
recurrent AB
theory
5
4
3
2
1
-1.0
-0.5
0.0
0.5
phase ϕ (π)

# Mathematical model – Focusing NLS

We study the anomalous waves on the focusing NLS equation (SfNLS) with periodic boundary conditions:

$$iu_t + u_{xx} + 2u^2\bar{u} = 0$$

We use the following Cauchy data (anomalous waves Cauchy problem):

$$u(x,0) = a + \epsilon v(x), \ \ v(x+L) \equiv v(x), \ \ |\epsilon| \ll 1,$$

$$v(x) = \sum_{j\geq 1}\left(c_j e^{ik_j x} + c_{-j}e^{-ik_j x}\right), \ \ k_j = \frac{2\pi}{L}j, \ \ |c_j| = O(1),$$

To simplify calculations we also assume that the period $L$ is generic: $L \neq \pi n$, $n \in \mathbb{Z}$.

# Unstable modes

The unstable background: ($\epsilon = 0$):

$$u_0(x,t) \; = \; a e^{2i|a|^2 t}.$$

The first $N$ harmonics are unstable, where

$$N = \left[ \frac{|a|L}{\pi} \right]$$

with the growing factors in the linear mode are:

$$\sigma_j = |a| k_j \sqrt{4|c_0|^2 - k_j^2}, \;\; 1 \leq j \leq N,$$

All other modes are stable. They give only small corrections and we discard them.

**We assume:** $\pi/|a| < L < 2\pi/|a|$ i.e. we have exactly one unstable mode. Therefore:

$$u(x,0) = a\left(1 + \epsilon\Big(c_1 e^{k_1 x} + c_{-1} e^{-ik_1 x}\Big)\right), \;\; k_1 = \frac{2\pi}{L}, \;\; \epsilon \ll 1,$$

The unstable mode is described by Riemann theta functions of 2 variables. **They are rather complicated.**

But for this special Cauchy data it admits a good approximation as a sequence of Akhmediev breathers (Grinevich–Santini).

**Akhmediev breathers:**

N. N. Akhmediev, V. M. Eleonskii, and N. E. Kulagin, "Exact first order solutions of the Nonlinear Schdinger equation", Theor. Math. Phys, 72, 809 (1987).

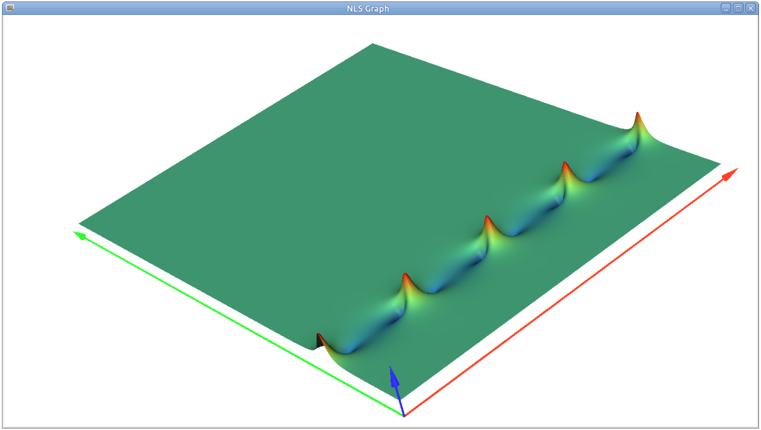
$$\mathcal{A}(x,t;\theta,X,T) =$$

$$= a\, e^{2i|a|^2 t} \cdot \frac{\cosh[\sigma(\theta)(t-T)+2i\theta] + \sin\theta\cos[k(\theta)(x-X)]}{\cosh[\sigma(\theta)(t-T)] - \sin\theta\cos[k(\theta)(x-X)]},$$

$$k_1 = k(\theta) = 2|a|\cos\theta,\ \sigma(\theta) = k(\theta)\sqrt{4|a|^2 - k^2(\theta)} = 2|a|^2\sin(2\theta),$$

# Akhmediev breathers

They are spatially periodic and localized in time:




The $x$ coordinate axis marked red, the $t$ coordinate axis marked green. In the future we draw only one period of solution with respect to $x$.

# One unstable mode

Approximation of the genus 2 solution:

$$u(x,t) = \sum_{m=0}^{n} \mathcal{A}\Big(x,t;\phi_1,x^{(m)},t^{(m)}\Big)e^{i\rho^{(m)}} - \frac{1-e^{4in\phi_1}}{1-e^{4i\phi_1}} a e^{2i|a|^2 t}, \quad x\in[0,L],$$

with the following parameters, expressed in terms of elementary functions:

$$x^{(m)} = X^{(1)} + (m-1)\Delta X, \quad t^{(m)} = T^{(1)} + (m-1)\Delta T,$$

$$X^{(1)} = \frac{\arg\alpha}{k_1} + \frac{L}{4}, \quad \Delta X = \frac{\arg(\alpha\beta)}{k_1}, \quad (\bmod\ L),$$

$$T^{(1)} \equiv \frac{1}{\sigma_1}\log\left(\frac{\sigma_1^2}{2|a|^4\epsilon|\alpha|}\right), \quad \Delta T = \frac{1}{\sigma_1}\log\left(\frac{\sigma_1^4}{4|a|^8\epsilon^2|\alpha\beta|}\right),$$

$$\rho^{(m)} = 2\phi_1 + (m-1)4\phi_1, \quad n = \left\lfloor \frac{T-T^{(1)}}{\Delta T} + \frac{1}{2}\right\rfloor,$$

$$\cos\phi_1 = \frac{\pi}{L|a|}, \quad \alpha = e^{-i\phi_1}\overline{c_1} - e^{i\phi_1}c_{-1}, \quad \beta = e^{i\phi_1}\overline{c_{-1}} - e^{-i\phi_1}c_1,$$

# One unstable mode

The spectra curve has genus $g = 2$ and 6 branch points: $E_0$, $E_1$, $E_2$, $\bar{E}_0$, $\bar{E}_1$, $\bar{E}_2$. The pair $E_1$, $E_2$ is obtained as a results of splitting the resonant point $\lambda_1 = i|a|\sin\phi_1$:

$$E_l = \lambda_1 + (-1)^l \frac{\epsilon |a|^2}{2\lambda_1}\sqrt{\alpha\beta} + O(\epsilon^2), \quad l = 1, 2,$$

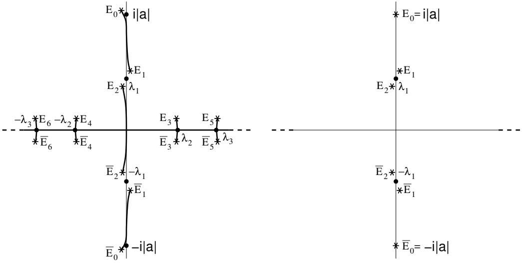



Figure: Right: the exact spectrum; Left: the approximating curve.

# One unstable mode

Generic solutions correspond to “long tori”.
The Akhmediev breather corresponds to the rational curve:

$$E_1 = E_2, \quad \bar{E}_1 = \bar{E}_2.$$

For Akhmediev breather we have a **homoclinic (whiskered) torus**:

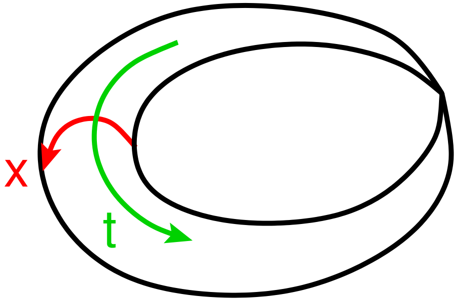



Figure: *x*-dynamics corresponds to the motion along the short cycle, *t*-dynamics corresponds to the motion along the infinite cycle

# Effect of small loss/gain

The first appearance of Akhmediev breather is very stable. In contrast, the recurrence is very sensitive to perturbations of Cauchy data or equations.

Effect of small loss/gain

$$iu_t + u_{xx} + 2u^2\bar{u} = -i\nu u, \quad u = u(x,t), \quad \nu \in \mathbb{R}, \quad |\nu| \ll 1.$$

was recently analytically studied in:

Coppini F., Grinevich P.G., Santini P.M. “The effect of a small loss or gain in the periodic NLS anomalous wave dynamics. I” - Phys. Rev. E, 2020, v.101, No 3, 032204, 8 pages, - Published 6 March 2020; doi:10.1103/PhysRevE.101.032204.

# Samll loss/gain – experiments and numerics:

Our aim was to explain the results of experimental and numerical observations:

O. Kimmoun, H.C. Hsu, H. Branger, M.S. Li, Y.Y. Chen, C. Kharif, M. Onorato, E.J.R. Kelleher, B. Kibler, N. Akhmediev, A. Chabchoub, "Modulation Instability and Phase-Shifted Fermi-Pasta-Ulam Recurrence", *Scientific Reports*, **6**, Article number: 28516 (2016), doi:10.1038/srep28516.

J.M. Soto-Crespo, N. Devine, and N. Akhmediev, "Adiabatic transformation of continuous waves into trains of pulses", *PHYSICAL REVIEW A*, **96**, 023825 (2017).

# Generic initial data:

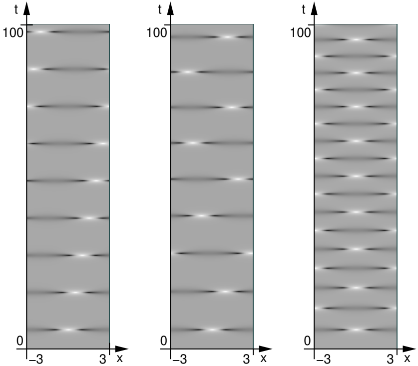



Figure: $-L/2 \le x \le L/2$, $0 \le t \le 100$, $L = 6$, $\epsilon = 10^{-4}$, generic initial data: $c_1 = 0.5$ and $c_{-1} = 0.15 - 0.2i$. Form left to right: $\nu = 0$, $\nu = 10^{-9}$, and $\nu = 10^{-5}$. The first appearance is essentially the same in all the three cases.

# Symmetric initial data:

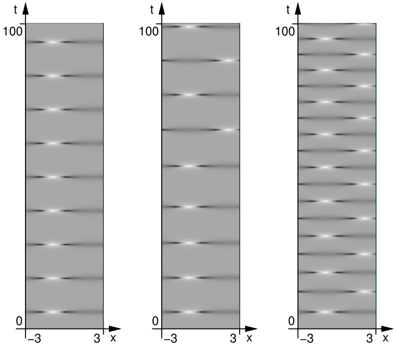



Figure: The density plot of $|u(x,t)|$ with $-L/2 \leq x \leq L/2$, $0 \leq t \leq 100$, $L = 6$, $\epsilon = 10^{-4}$, for a real initial condition ($c_{-j} = \overline{c_j}$, $\forall j$), with $c_1 = 0.3 + 0.4i$. Consequently $\alpha\beta > 0$. Left picture: $\nu = 0$, then $\Delta X = 0$. Center picture: $\nu = 10^{-9}$; then for $\tilde{m} = 6$, $Q_m$ changes its sign, from positive to negative values; correspondingly, $\Delta X_m$ switches from 0 to $L/2$. Right picture: $\nu = 10^{-5}$; then all $Q_m$ are negative and $\Delta X_m = L/2$ $\forall m$. The first appearance is essentially the same in all the three cases.

# Strong and weak losses.

The effect of strong losses was discussed in :
H. Segur, D. Henderson, J. Carter, J. Hammack, C.-M. Li, D. Pheiff, and K. Socha, “Stabilizing the Benjamin-Feir instability”, *J. Fluid Mech.* 539, 229 (2005).

The background is unstable if:

$$\cos\phi_1 = \frac{\pi}{L|a|}, \quad \Rightarrow \quad \left|\frac{L|a|}{\pi}\right| > 1.$$

If $a$ decays fast enough, at some moment the background become stable.

We were interested in the opposite situation: $\nu \sim \epsilon^2$ and $|a|$ is almost connstant, but the recurrence **changes essentially.**

# Analytic formulas.

We have the following approximate formulas *the spectral curve is not time–invariant, but it changes each time we have an anomalous wave:*

$$(E_1 - E_2)^2\Big|_{t=0} = -\frac{\epsilon^2|a|^2\alpha\beta}{\sin^2\phi_1},$$

$$(E_1^{(m)} - E_2^{(m)})^2 = -\frac{\epsilon^2|a|^2\alpha\beta}{\sin^2\phi_1} + 4m\nu\cot\phi_1, \quad m \geq 0,$$

*where* $E_1^{(m)}, E_2^{(m)}$ *are the branch points after the* $m^{th}$*-th breather.*
Therefore:

$$\Delta X_m := \tilde{x}^{(m+1)} - \tilde{x}^{(m)} = \frac{\arg(Q_m)}{k_1} \pmod{L},$$
$$\Delta T_m := \tilde{t}^{(m+1)} - \tilde{t}^{(m)} = \frac{1}{\sigma_1}\log\left(\frac{\sigma_1^4}{4\epsilon^2|Q_m|}\right),$$

$$\epsilon^2 Q_m = \epsilon^2\alpha\beta - \frac{\nu\sigma_1}{|a|^4}m, \quad m \geq 1, \tag{1}$$

# Evolution of the branch points

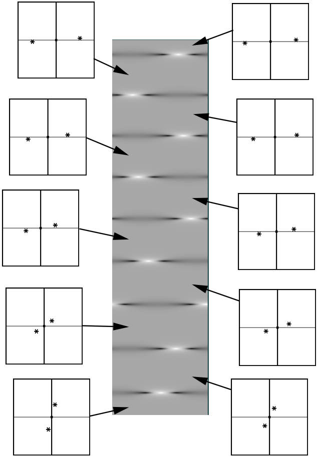
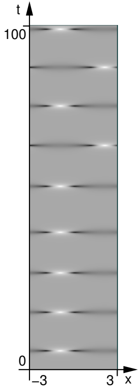


Figure: Evolution of the branch points

# Symmetric initial data numerics vs analytics:




$\tilde{t}^{(1)} = 5.51209$ (theory) $\tilde{t}^{(1)} = 5.51208$ (numerics)
$\Delta T_1 = 11.18230$ (theory) $\Delta T_1 = 11.18230$ (numerics)
$\Delta T_2 = 11.40337$ (theory) $\Delta T_2 = 11.40338$ (numerics);
$\Delta T_3 = 11.77375$ (theory) $\Delta T_3 = 11.77376$ (numerics);
$\Delta T_4 = 13.31847$ (theory) $\Delta T_4 = 13.31848$ (numerics);
$\Delta T_5 = 11.84989$ (theory) $\Delta T_5 = 11.84988$ (numerics);
$\Delta T_6 = 11.44140$ (theory) $\Delta T_6 = 11.44142$ (numerics);
$\Delta T_7 = 11.20765$ (theory) $\Delta T_7 = 11.20766$ (numerics);
$\Delta T_8 = 11.04319$ (theory) $\Delta T_8 = 11.04320$ (numerics)

# Linear perturbation theory near Akhmediev breather

Linear perturbation stability of Akhmediev breathers was studied in:

A. Calini, C.M. Schober, "Dynamical criteria for rogue waves in nonlinear Schrödinger models", *Nonlinearity*, **25**:12 (2012) R99–R116; doi:10.1088/0951-7715/25/12/R99.

A. Calini, C.M. Schober, "Observable and reproducible rogue waves", J. Opt. 15 (2013) 105201 (9pp).

A. Calini, C.M. Schober, "Numerical investigation of stability of breather-type solutions of the nonlinear Schrödinger equation", Nat. Hazards Earth Syst. Sci., 14, 14311440, 2014 www.nat-hazards-earth-syst-sci.net/14/1431/2014/doi:10.5194/nhess-14-1431-2014.

# Linear perturbation theory near Akhmediev breather

To simplify formulas we use the following gauge transformation.

$$u(x,t) \to \exp(2it)u(x,t), \quad \vec{\psi} \to \exp(i\sigma_3 t)\vec{\psi}, \text{ and}$$

$$iu_t + u_{xx} + 2|u|^2 u - 2u = 0.$$

Let $\vec{\phi} = \begin{pmatrix} \phi_1(\lambda,x) \\ \phi_2(\lambda,x) \end{pmatrix}$, $\vec{\psi} = \begin{pmatrix} \psi_1(\lambda,x) \\ \psi_2(\lambda,x) \end{pmatrix}$ be Lax pair eigenfuntions

with the same $\lambda$. Then the squared eigenfinctions:

$$\left\langle \vec{\psi}(\lambda,x,t), \vec{\varphi}(\lambda,x,t) \right\rangle_+ := \psi_1(\lambda,x,t)\varphi_1(\lambda,x,t) + \overline{\psi_2(\lambda,x,t)\varphi_2(\lambda,x,t)},$$
$$\left\langle \vec{\psi}(\lambda,x,t), \vec{\varphi}(\lambda,x,t) \right\rangle_- := i\left[\psi_1(\lambda,x,t)\varphi_1(\lambda,x,t) - \overline{\psi_2(\lambda,x,t)\varphi_2(\lambda,x,t)}\right]$$

satisfy the linearized NLS equation

$$iw_t + w_{xx} + 4|u|^2 w + 2u^2\bar{w} - 2w = 0.$$

# Linear perturbation theory near Akhmediev breather

In the aforementioned papers it was shown that if we have $N$ unstable modes and nonlinear superpositions of $M$ Akhmediev breathers, $M \leq N$ then

1. If not all unstable modes are exited ($M < N$) then there exist $x$-periodic squared eigenfunctions exponentially growing in $t$;
2. If all unstable modes are exited ($M = N$) then all $x$-periodic with the period $L$ squared eigenfunctions are bounded in $t$;

Therefore the following conclusion was made:

1. If not all unstable modes are exited ($M < N$), the solution is unstable;
2. If all unstable modes are exited ($M = N$) then one has "saturation of instabilities".

But the second conclusion contradicts our results, because **small perturbations generate recurrence.**

# Linear perturbation theory near Akhmediev breather

We obtained the following resolution of the paradox (we studied the case $M = N = 1$):
P.G. Grinevich, P.M. Santini, "The linear and nonlinear instability of the Akhmediev breather", arXiv:2011.11402.

Due to presence of non-removable double points the spectral decomposition of linerized NLS solutions includes not only $x$-periodic squared eigenfunctions, but also some special combinations of derivatives with respect to the spectral parameter.

Let us demonstrate the "missed modes".

# The “missed modes”

For $u_0 = 1$ we use the following basis of eigenfunctions:

$$\vec{\psi}_0^{\pm}(\lambda, x, t) = \begin{bmatrix} \sqrt{\mu \mp \lambda} \\ \pm\sqrt{\mu \pm \lambda} \end{bmatrix} e^{\pm\theta}, \quad \theta = i\mu x + 2i\mu\lambda t, \quad \mu^2 = \lambda^2 + 1.$$

Let us denote

$$\vec{q}(\lambda) = \begin{bmatrix} q_1(\lambda) \\ q_2(\lambda) \end{bmatrix} = \begin{bmatrix} \sqrt{\mu - \lambda}\, e^{\theta(\lambda)} + \sqrt{\mu + \lambda}\, e^{-\theta(\lambda)} \\ \sqrt{\mu + \lambda}\, e^{\theta(\lambda)} - \sqrt{\mu - \lambda}\, e^{-\theta(\lambda)} \end{bmatrix},$$

$$\vec{r}(\lambda) = \begin{bmatrix} r_1(\lambda) \\ r_2(\lambda) \end{bmatrix} = \begin{bmatrix} \sqrt{\mu - \lambda}\, e^{\theta(\lambda)} - \sqrt{\mu + \lambda}\, e^{-\theta(\lambda)} \\ \sqrt{\mu + \lambda}\, e^{\theta(\lambda)} + \sqrt{\mu - \lambda}\, e^{-\theta(\lambda)} \end{bmatrix},$$

$$\vec{\phi}(\lambda) = \begin{bmatrix} \phi_1(\lambda) \\ \phi_2(\lambda) \end{bmatrix} = \begin{bmatrix} 1 \\ (\mu + \lambda) \end{bmatrix} e^{\theta(\lambda)}$$

The unstable modes correspond to the pure imaginary part of the spectrum:

$$\mu \in \mathbb{R}, \quad \lambda \in i\mathbb{R}, \ |\lambda| \le 1.$$

# The “missed modes”

The resonant point is:

$$\lambda_1 = \sqrt{\mu_1^2 - 1}, \quad \mu_1 = \frac{k_1}{2} = \frac{\pi}{L},$$

$$k = k_1 = 2\mu_1, \quad \sigma = \sigma_1 = -4i\lambda_1\mu_1, \quad \theta(\lambda_1) = \frac{1}{2}\left(ikx - \sigma t\right).$$

Denote

$$\vec{q} = \vec{q}(\lambda_1), \quad \vec{r} = \vec{r}(\lambda_1),$$

The Darboux transformation operator is defined by:

$$\mathfrak{D} = (\lambda - \lambda_1)E + \frac{2\lambda_1}{|q_1|^2 + |q_2|^2}\begin{bmatrix} -\overline{q_2} \\ \overline{q_1} \end{bmatrix}[-q_2, q_1],$$

Operator $\mathfrak{D}$ maps the background eigenfunctions to the eigenfunctions for the Akhmediev breather.

$$\vec{\psi}(\lambda) = \mathfrak{D}\vec{\psi}_0(\lambda)$$

# The “missed modes”

Here Akhmediev breather reads:

$$u(x,t) = \frac{(\lambda_1^2+\mu_1^2)\cosh(\sigma t) + i\lambda_1\sin(kx) + 2\mu_1\lambda_1\sinh(\sigma t)}{\cosh(\sigma t) - i\lambda_1\sin(kx)}.$$

Let us denote

$$\vec{\chi}_+(\lambda) = \mathfrak{D}\vec{q}(\lambda),\ \ \vec{\chi}_-(\lambda) = \mathfrak{D}\vec{r}(\lambda),\ \ \tilde{\phi}(\lambda) = \mathfrak{D}\phi(\lambda),$$

$$f^{(n)}(x,t) := D_\mu^n f(\lambda,x,t)\Big|_{\lambda=\lambda_1},$$

where

$$D_\mu = \partial_\mu + \frac{\partial\lambda}{\partial\mu}\partial_\lambda = \partial_\mu + \frac{\mu}{\lambda}\partial_\lambda.$$

The last formula takes into account that $\lambda = \sqrt{\mu^2-1}$.

# The “missed modes”

The **real** derivatives of the squared eigenfunctions also satisfy linearized NSL. We consider the following combinations:

$$
\begin{aligned}
(D_\mu + D_{\bar{\mu}}) \left.\langle \chi_+(\lambda), \chi_-(\lambda) \rangle_\pm\right|_{\lambda=\lambda_1} &= \left\langle \chi_+^{(1)}, \chi_-^{(0)} \right\rangle_\pm , \\
(D_\mu + D_{\bar{\mu}})^2 \left.\langle \chi_+(\lambda), \chi_-(\lambda) \rangle_\pm\right|_{\lambda=\lambda_1} &= \left\langle \chi_+^{(2)}, \chi_-^{(0)} \right\rangle_\pm + 2 \left\langle \chi_+^{(1)}, \chi_-^{(1)} \right\rangle_\pm , \\
(D_\mu + D_{\bar{\mu}}) \left.\left\langle \tilde{\phi}(\lambda), \tilde{\phi}(\lambda) \right\rangle_\pm\right|_{\lambda=\lambda_1} &= 2 \left\langle \tilde{\phi}^{(1)}, \tilde{\phi}^{(0)} \right\rangle_\pm ,
\end{aligned}
$$

# The "missed modes"

**Main statement** The following combinations of the derivatives of the squared eigenfunctions with respect to the spectral parameter:

$$\mathrm{Sym}_1 = -2\lambda_1^2\Big[\mu_1\Big(\Big\langle\chi_+^{(2)},\chi_-^{(0)}\Big\rangle_+ + 2\Big\langle\chi_+^{(1)},\chi_-^{(1)}\Big\rangle_+\Big) - 4\Big\langle\chi_+^{(1)},\chi_-^{(0)}\Big\rangle_+ - 16\Big\langle\phi_+^{(1)},\phi_-^{(0)}\Big\rangle_+\Big],$$

$$\mathrm{Sym}_2 = -2\lambda_1^2\Big(\Big\langle\chi_+^{(2)},\chi_-^{(0)}\Big\rangle_- + 2\Big\langle\chi_+^{(1)},\chi_-^{(1)}\Big\rangle_- + \frac{2\mu_1}{\lambda_1^2}\Big\langle\chi_+^{(1)},\chi_-^{(0)}\Big\rangle_-\Big),$$

where have the following properties:

- They are $x$-periodic with period $L$;
- They exponentially grow as $t \to \pm\infty$;
- They are solutions of the linearized NLS near the Akhmediev breather.

Therefore they represent the "missed modes".

# The “missed modes”

## Maple calculation:

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> Ef11;

$$-\frac{1}{4}\Big(k\big(-16\sinh(\sigma t)\cos(3kx)k^2+8\sinh(\sigma t)\cos(3kx)k^4-\sinh(\sigma t)\cos(3kx)k^6-24\sinh(\sigma t)\cos(kx)k^4+\sinh(\sigma t)\cos(kx)k^6$$
$$-32\sigma\sinh(2\sigma t)k^2+8\sigma\sinh(2\sigma t)k^4+48\sinh(3\sigma t)\cos(kx)k^2-16\sinh(3\sigma t)\cos(kx)k^4+64\sinh(\sigma t)\cos(kx)k^2+64\,\mathrm{I}\,k\sin(2kx)$$
$$-48\,\mathrm{I}\,k^3\sin(2kx)+8\,\mathrm{I}\,k^5\sin(2kx)+64\,\mathrm{I}\cos(2kx)k^2-32\,\mathrm{I}\cos(2kx)k^4+4\,\mathrm{I}\cos(2kx)k^6-64\,\mathrm{I}\cosh(2\sigma t)k^2+48\,\mathrm{I}\cosh(2\sigma t)k^4$$
$$-8\,\mathrm{I}\cosh(2\sigma t)k^6+128\,\mathrm{I}\,k^2-48\,\mathrm{I}\,k^4+4\,\mathrm{I}\,k^6+32\sigma\sinh(2\sigma t)\sin(2kx)k-8\sigma\sinh(2\sigma t)\sin(2kx)k^3+64\,\mathrm{I}\,\sigma\cos(kx)\cosh(\sigma t)$$
$$+16\,\mathrm{I}\,\sigma\cosh(3\sigma t)\cos(kx)+64\,\mathrm{I}\,k\cosh(2\sigma t)\sin(2kx)-48\,\mathrm{I}\,k^3\cosh(2\sigma t)\sin(2kx)+8\,\mathrm{I}\,k^5\cosh(2\sigma t)\sin(2kx)-16\,\mathrm{I}\,\sigma\cosh(\sigma t)\cos(3kx)$$
$$+\mathrm{I}\,\sigma\cos(kx)\cosh(\sigma t)k^4-48\,t\sigma\cos(kx)\cosh(\sigma t)k^4+8\,t\sigma\cos(kx)\cosh(\sigma t)k^6-16\,\mathrm{I}\,\sigma\cosh(3\sigma t)\cos(kx)k^2-40\,\mathrm{I}\,\sigma\cos(kx)\cosh(\sigma t)k^2$$
$$-192\,\mathrm{I}\,t\cos(kx)\sinh(\sigma t)k^4+80\,\mathrm{I}\,t\cos(kx)\sinh(\sigma t)k^6+8\,\mathrm{I}\,\sigma\cosh(\sigma t)\cos(3kx)k^2-\mathrm{I}\,\sigma\cosh(\sigma t)\cos(3kx)k^4-8\,\mathrm{I}\,t\cos(kx)\sinh(\sigma t)k^8\big)\Big)\Big/$$
$$\big(4\cosh(\sigma t)^2k+4\sigma\sin(kx)\cosh(\sigma t)+4k-k^3-4\cos(kx)^2k+\cos(kx)^2k^3\big) \qquad (37)$$

> Ef21;

$$-\frac{1}{2}\Big(\mathrm{I}\,\sigma\sinh(\sigma t)\sin(3kx)k^4+208\,\sigma t\sinh(\sigma t)\sin(kx)k^4-24\,\sigma t\sinh(\sigma t)\sin(kx)k^6-256\,k^2\sigma t\sinh(\sigma t)\sin(kx)-8\,\mathrm{I}\,\sigma\sinh(\sigma t)\sin(3kx)k^2$$
$$+40\,\mathrm{I}\,\sigma\sinh(\sigma t)\sin(kx)k^2-3\,\mathrm{I}\,\sigma\sinh(\sigma t)\sin(kx)k^4+16\,\mathrm{I}\,\sigma\sinh(3\sigma t)\sin(kx)k^2-304\,\mathrm{I}\,k^6t\cosh(\sigma t)\sin(kx)+24\,\mathrm{I}\,k^8t\cosh(\sigma t)\sin(kx)$$
$$+128\,\mathrm{I}\,\sigma t\cosh(2\sigma t)k^3-96\,\mathrm{I}\,\sigma t\cos(2kx)k^3-32\,\mathrm{I}\,\sigma t\cosh(2\sigma t)k^5+16\,\mathrm{I}\,\sigma t\cos(2kx)k^5+1088\,\mathrm{I}\,k^4t\cosh(\sigma t)\sin(kx)$$
$$-1024\,\mathrm{I}\,k^2t\cosh(\sigma t)\sin(kx)+128\,\mathrm{I}\,\sigma t\cos(2kx)k-128\,\mathrm{I}\,\sigma t\cosh(2\sigma t)k-64\,\mathrm{I}\sinh(2\sigma t)k+32\sigma\cosh(2\sigma t)\cos(2kx)k$$
$$-8\sigma\cosh(2\sigma t)\cos(2kx)k^3-256\,\mathrm{I}\,\sigma t k+64\,\mathrm{I}\sinh(2\sigma t)\cos(2kx)k+16\,\mathrm{I}\,\sigma\sinh(\sigma t)\sin(3kx)+256\,\mathrm{I}\,k^3t\sigma-32\,\mathrm{I}\,k^5t\sigma$$
$$-48\,\mathrm{I}\sinh(2\sigma t)\cos(2kx)k^3+8\,\mathrm{I}\sinh(2\sigma t)\cos(2kx)k^5-64\,\mathrm{I}\,\sigma\sinh(\sigma t)\sin(kx)-16\,\mathrm{I}\,\sigma\sinh(3\sigma t)\sin(kx)-64\sigma k$$
$$+16k^2\cosh(\sigma t)\sin(3kx)+32\sigma\cos(2kx)k-64\sigma\cosh(2\sigma t)k-320\sin(kx)k^2\cosh(\sigma t)-48k^2\cosh(3\sigma t)\sin(kx)+56k^4\sin(kx)\cosh(\sigma t)$$
$$-3\sin(kx)k^6\cosh(\sigma t)-8k^4\cosh(\sigma t)\sin(3kx)+k^6\cosh(\sigma t)\sin(3kx)+16k^4\cosh(3\sigma t)\sin(kx)-8\sigma\cos(2kx)k^3-256\,t\sinh(2\sigma t)k^3$$
$$+192\,t\sinh(2\sigma t)k^5-32\,t\sinh(2\sigma t)k^7+64\,\mathrm{I}\,k^3\sinh(2\sigma t)\Big)\Big/\big(4\cosh(\sigma t)^2k+4\sigma\sin(kx)\cosh(\sigma t)+4k-k^3-4\cos(kx)^2k+\cos(kx)^2k^3\big) \qquad (38)$$

Memory: 1789.92M | Time: 733.71s | Math Mode

# The “missed modes”

## Linearized NLS substitution:

```
> 
> 
> u1;
```

$$\frac{-\frac{1}{2}\,\frac{\sigma\sin(k\,x)}{k}+\cosh(\sigma\,t)+\frac{1}{2}\,I\,\sigma\sinh(\sigma\,t)-\frac{1}{2}\,\frac{\sigma^2\cosh(\sigma\,t)}{k^2}}{\cosh(\sigma\,t)+\frac{1}{2}\,\frac{\sigma\sin(k\,x)}{k}} \qquad (45)$$

```
> u1b:
> 
> 
> Lin := Ef11:
> Linb := subs(subc, Lin):
> LinNLS := I·diff(Lin, t) + diff(Lin, x, x) + 4·u1·u1b·Lin + 2·u1·u1·Linb − 2·Lin:
> Eq2 := simplify(subs(subk, LinNLS)):
> Eq21 := simplify(convert(numer(Eq2), trig)) : Eq22 := simplify(subs(subk, Eq21)):
> Eq23 := convert(Eq22, exp) : Eq24 := simplify(expand(subs(subk, Eq23))):
> Eq24;
```

$$0 \qquad (46)$$

```
> 
> 
> 
> Lin := Ef21:
> Linb := subs(subc, Lin):
> LinNLS := I·diff(Lin, t) + diff(Lin, x, x) + 4·u1·u1b·Lin + 2·u1·u1·Linb − 2·Lin:
> Eq2 := simplify(subs(subk, LinNLS)):
> Eq21 := simplify(convert(numer(Eq2), trig)) : Eq22 := simplify(subs(subk, Eq21)):
> Eq23 := convert(Eq22, exp) : Eq24 := simplify(expand(subs(subk, Eq23))):
> Eq24;
```

$$0 \qquad (47)$$

```
> 
> 
```

# The "missed modes"

Let us shift the spatial varibale $x \to x - L/4$. We obtain:

$$u(x,t) = \frac{(2k^2 - \sigma^2)\cosh(\sigma t) + ik^2\sigma\sinh(\sigma t) + k\sigma\cos(kx)}{k(2\cosh(\sigma t)k - \sigma\, cos(kx))}$$

The even part of $\widehat{\mathrm{Sym}}_1$ is bounded in $t$. Denote by $\widehat{\mathrm{Sym}}_1$ the odd part of $\mathrm{Sym}_1$. We have

$$\widehat{\mathrm{Sym}}_1(x,t) = \frac{1}{2}\left(\mathrm{Sym}_1(x,t) - \mathrm{Sym}_1(-x,t)\right)$$

$$\widehat{\mathrm{Sym}}_1(x,t) = k\frac{\widehat{\mathrm{Num}}_1(x,t)}{\mathcal{D}(x,t)}$$

Solution $\mathrm{Sym}_2$ becomes even in $x$, and reads

$$\mathrm{Sym}_2(x,t) = \frac{\mathrm{Num}_2(x,t)}{\mathrm{Den}(x,t)}$$

# The "missed modes"

$$\widehat{\mathrm{Num}}_1(x,t) = \Big\{\Big[48\sigma k^4 - 8\sigma k^6\Big]\,\cosh(\sigma t) + \Big[192ik^4 + 8ik^8 - 80ik^6\Big]\,\sinh(\sigma t)\Big\}\, t\sin(kx) +$$

$$+\Big\{\Big[8i\sigma k^2 - i\sigma k^4 - 16i\sigma\Big]\,\cosh(\sigma t) + \Big[8k^4 - 16k^2 - k^6\Big]\,\sinh(\sigma t)\Big\}\,\sin(3kx) +$$

$$+\Big\{\Big[-48ik^3 + 64ik + 8ik^5\Big]\cosh(2\sigma t) + \Big[32\sigma k - 8\sigma k^3\Big]\sinh(2\sigma t) +$$

$$+\Big[8ik^5 - 48ik^3 + 64ik\Big]\Big\}\,\sin(2kx) +$$

$$+\Big\{\Big[-16i\sigma + 16i\sigma k^2\Big]\,\cosh(3\sigma t) + \Big[-64i\sigma + 40i\sigma k^2 - ik^4\sigma\Big]\,\cosh(\sigma t) +$$

$$+\Big[16k^4 - 48k^2\Big]\,\sinh(3\sigma t) + \Big[-64k^2 - k^6 + 24k^4\Big]\,\sinh(\sigma t)\Big\}\,\sin(kx),$$

$$\mathrm{Den}(x,t) = 4\,\Big[4\,k\cosh^2(\sigma\, t) - 4\,\sigma\,\cosh(\sigma\, t)\,\cos(k\,x) + k(4-k^2)\cos^2(kx)\Big]$$

# The “missed modes”

$$\mathrm{Num}_2(x,t) = \left\{256 i\sigma k - 192 i k^3\sigma + 32 i k^5\sigma\right\} t\cos(2kx) +$$
$$+ \left\{\left[2176 i k^4 + 48 i k^8 - 2048 i k^2 - 608 i k^6\right] \cosh(\sigma t) + \left[416\sigma k^4 - 48\sigma k^6 - 512\sigma k^2\right] \sinh(\sigma t)\right\} t\cos(kx) -$$
$$- 64k\left\{\left[6k^4 - 8k^2 - k^6\right] \sinh(2\sigma t) + \left[4 i\sigma k^2 - i k^4\sigma - 4 i\sigma\right] \cosh(2\sigma t) + \left[- i k^4\sigma + 8 i\sigma k^2 - 8 i\sigma\right]\right\} t +$$
$$+ \left\{\left[-2k^6 - 32k^2 + 16k^4\right] \cosh(\sigma t) + \left[-2 i\sigma k^4 + 16 i\sigma k^2 - 32 i\sigma\right] \sinh(\sigma t)\right\} \cos(3kx) +$$
$$+ \left\{\left[64\sigma k - 16\sigma k^3\right] \cosh(2\sigma t) + \left[-96 i k^3 + 128 i k + 16 i k^5\right] \sinh(2\sigma t) + \left[64\sigma k - 16\sigma k^3\right]\right\} \cos(2kx) +$$
$$+ \left\{\left[-96k^2 + 32k^4\right] \cosh(3\sigma t) + \left[-640k^2 + 112k^4 - 6k^6\right] \cosh(\sigma t) +\right.$$
$$\left. + \left[32 i\sigma k^2 - 32 i\sigma\right] \sinh(3\sigma t) + \left[-6 i\sigma k^4 + 80 i\sigma k^2 - 128 i\sigma\right] \sinh(\sigma t)\right\} \cos(kx) -$$
$$- 64k\left\{\left[2 i k^2 - 2i\right] \sinh(2\sigma t) - 2\sigma\cosh(2\sigma t) - 2\sigma\right\},$$

$$\mathrm{Den}(x,t) = 4\left[4\, k\cosh^2(\sigma\, t) - 4\,\sigma\, \cosh(\sigma\, t)\, \cos(k\, x) + k(4-k^2)\cos^2(kx)\right].$$