

Nonlinear Spectral Synthesis of Soliton Gas

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Soliton solution of the NLSE

Nonlinear Schrodinger equation (NLSE)

$$i\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = 0$$

Zakharov-Shabat scattering problem:

$$\Phi_x = \begin{pmatrix} -i\lambda & \psi \\ -\psi^* & i\lambda \end{pmatrix} \Phi = Q\Phi = 0$$

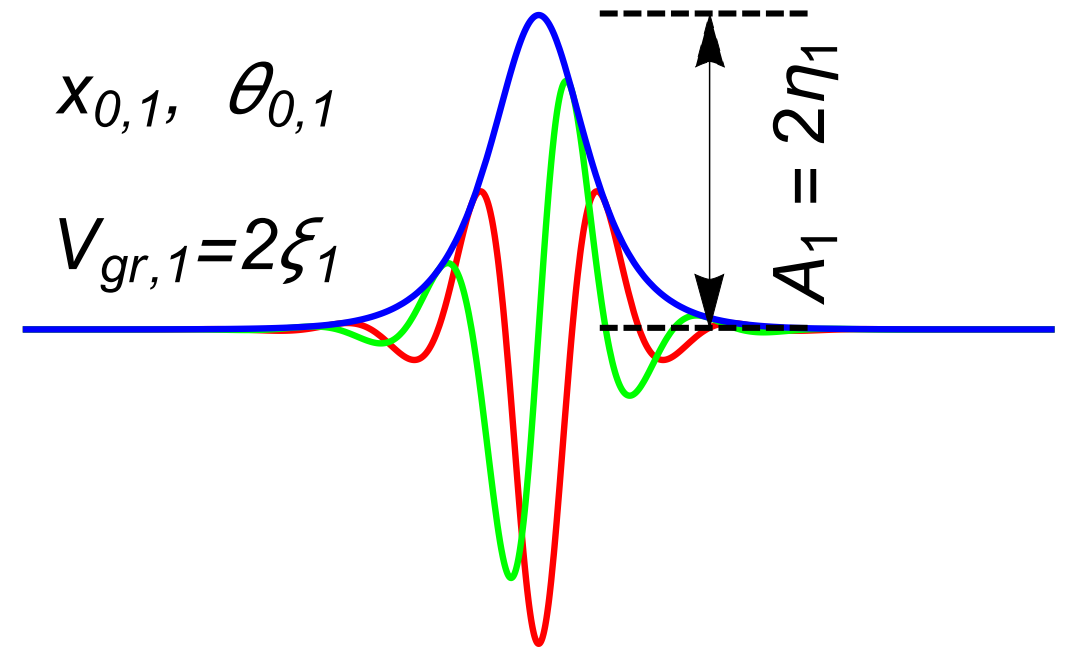
$$\hat{L} = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial x} - i \begin{pmatrix} 0 & \psi \\ \psi^* & 0 \end{pmatrix}$$

$$\hat{L}\Phi = \lambda \Phi$$

Soliton complex discrete eigenvalue:

$$\lambda_j = \xi_j + i\eta_j$$

$$\psi_{1SS}(x, t) = 2\eta \frac{\exp[-2i\xi(x - x_0) - 2i(\xi^2 - \eta^2)t + i\theta]}{ch[2\eta(x - x_0) + 4\xi\eta t]}$$

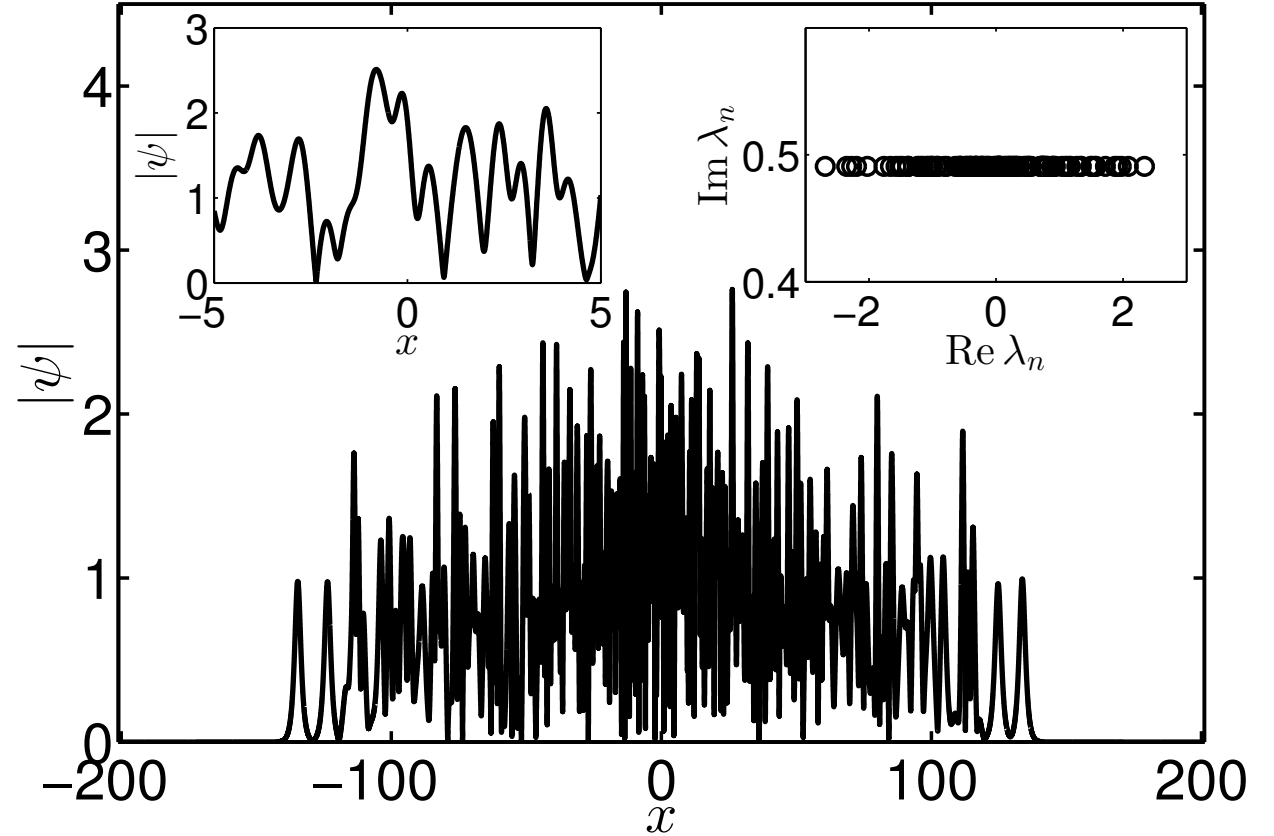


Inverse scattering transform for multi-soliton wave fields

$$\psi_{NSS}(x, t) = \frac{\det \begin{bmatrix} 0 & q_{1,2} & \cdots & q_{n,2} \\ q_{1,1}^* & \frac{(q_1 \cdot q_1^*)}{\lambda_1 + \lambda_1^*} & \cdots & \frac{(q_1 \cdot q_n^*)}{\lambda_1 + \lambda_n^*} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n,1}^* & \frac{(q_n \cdot q_1^*)}{\lambda_n + \lambda_1^*} & \cdots & \frac{(q_n \cdot q_n^*)}{\lambda_n + \lambda_n^*} \end{bmatrix}}{\det \begin{bmatrix} \frac{(q_1 \cdot q_1^*)}{\lambda_1 + \lambda_1^*} & \cdots & \frac{(q_1 \cdot q_n^*)}{\lambda_1 + \lambda_n^*} \\ \vdots & \ddots & \vdots \\ \frac{(q_n \cdot q_1^*)}{\lambda_n + \lambda_1^*} & \cdots & \frac{(q_n \cdot q_n^*)}{\lambda_n + \lambda_n^*} \end{bmatrix}}$$

$$q_j = (e^{-\phi_j}, e^{+\phi_j})^T$$

$$\phi_j = -i\lambda_j (x - x_{0,j}) - i\lambda_j^2 t - i\theta_j/2$$



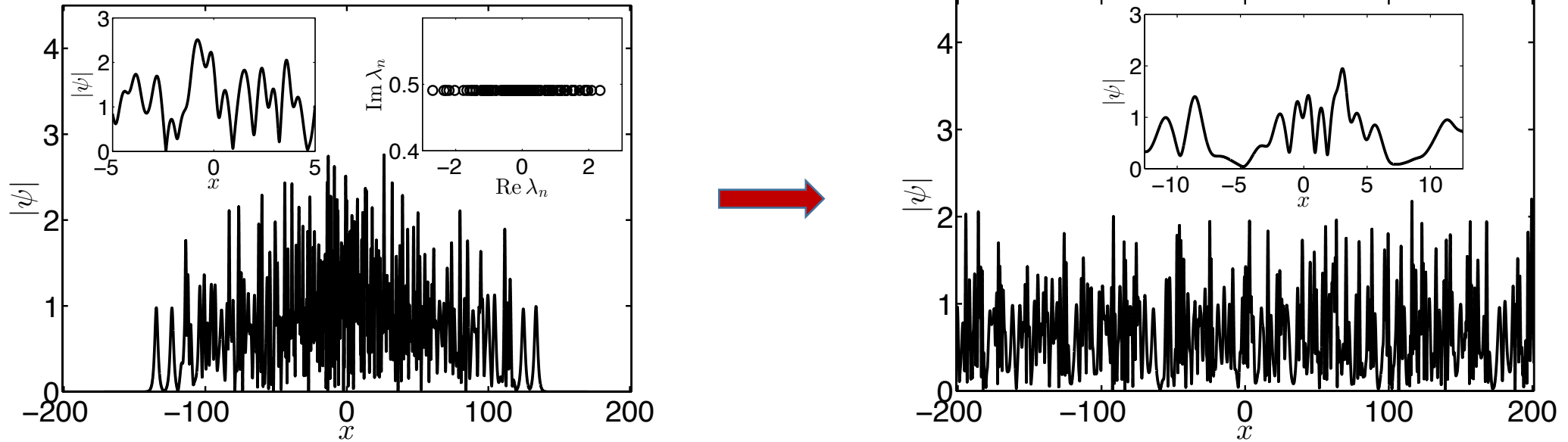
128-soliton solution with random phases

We need high precision !

$$x_{0,k} \sim \text{Random}[-2, 2]$$

Statistically homogeneous soliton gas

The question: what is the distribution of $x_{0,k}$ for homogeneous soliton gas??



The same 128-SS as before after stochastization during simulations with periodic boundary conditions (simulation time ~ 100).

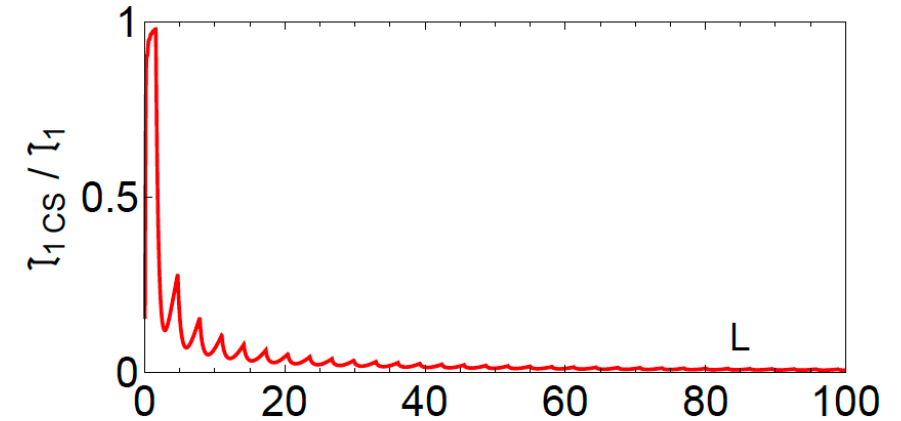
Can the mutual soliton shifts be relevant?
$$\Delta x_{0,k} = \frac{1}{4\eta_k} \ln \left(\frac{|\lambda_j - \lambda_k^*|^2}{|\lambda_j - \lambda_k|^2} \right)$$

The Bohr-Sommerfeld quantization rule for the semiclassical ZS problem:

$$\int \sqrt{|\psi|^2 - \eta_n^2} dx = 2\pi(n - 1/2), \quad \lambda_n = \xi_n + i\eta_n, \quad \xi_n/\eta_n \sim N^{-1/2}$$



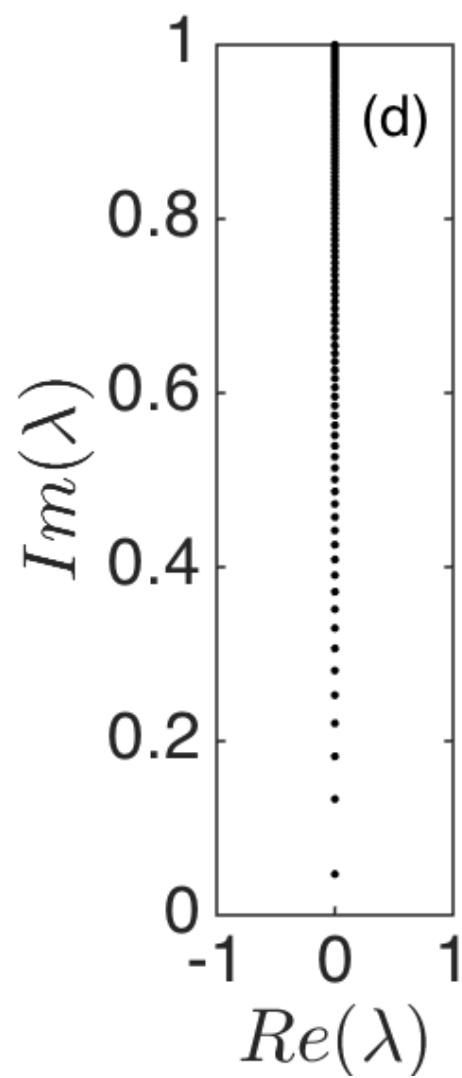
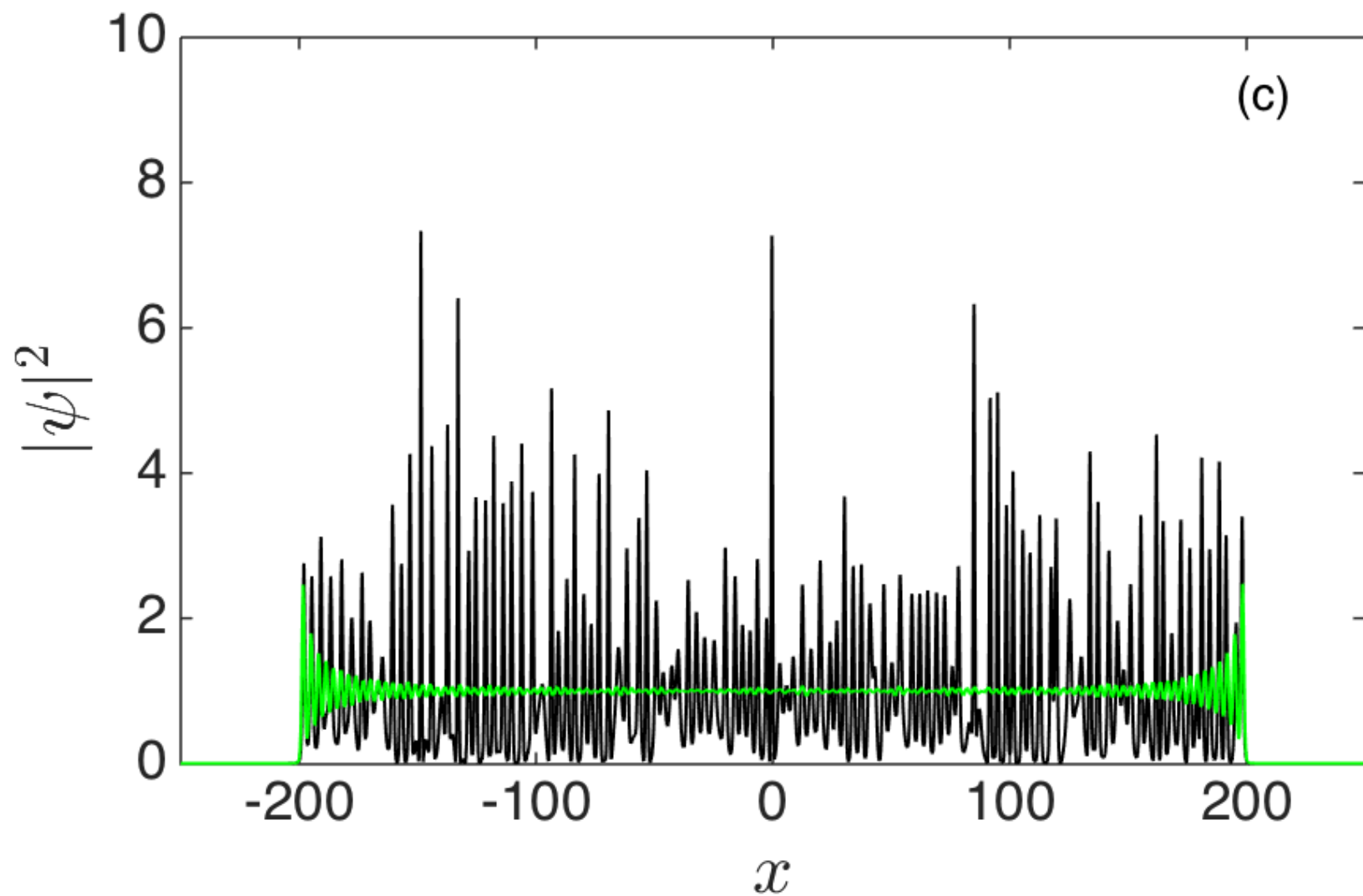
Spatially large region of the condensate, which we model by a rectangular function.



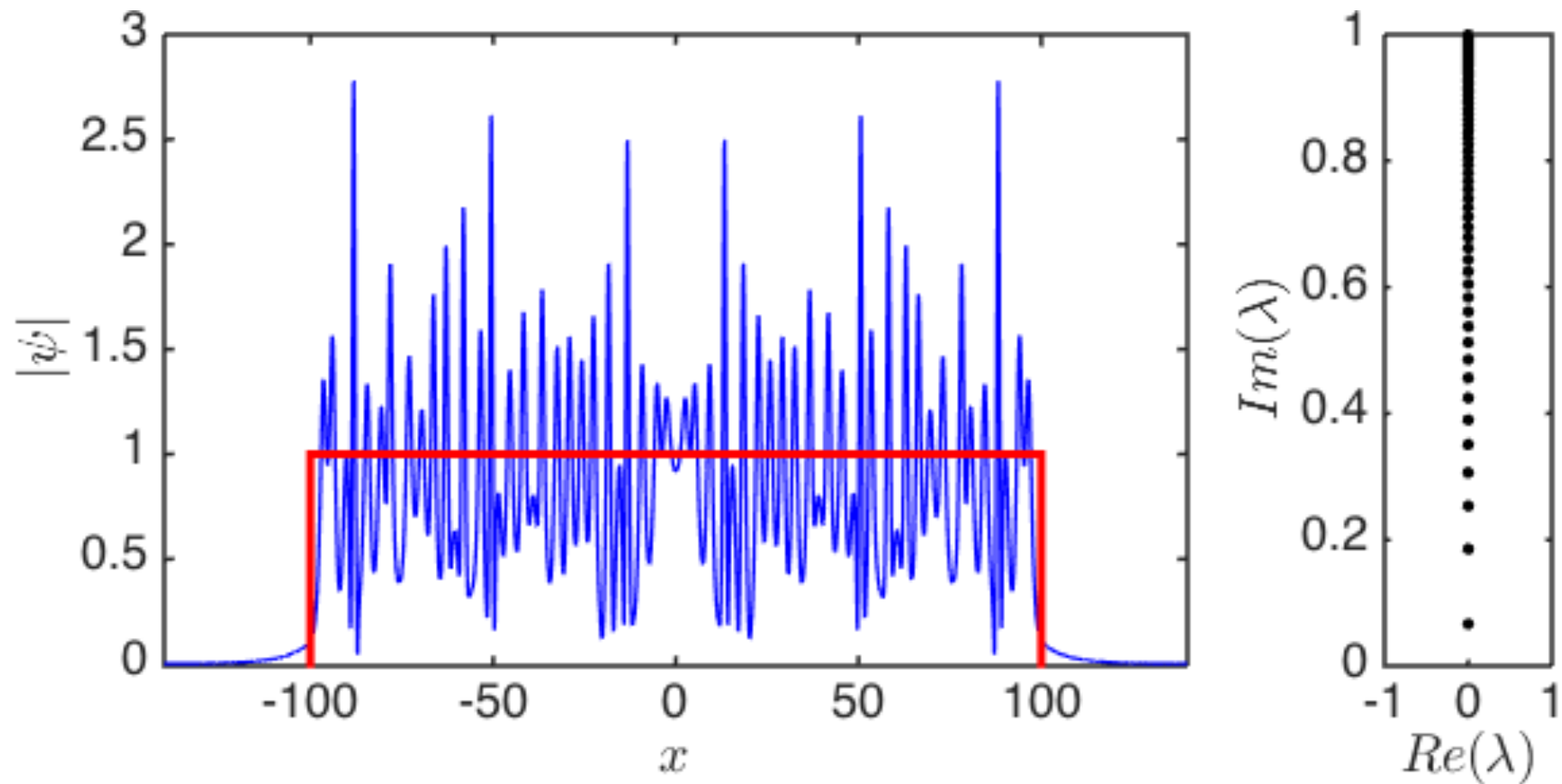
Relative impact of continuous spectrum to the first integral of the rectangular wave field.

Semiclassical distribution of soliton eigenvalues in the box potential:

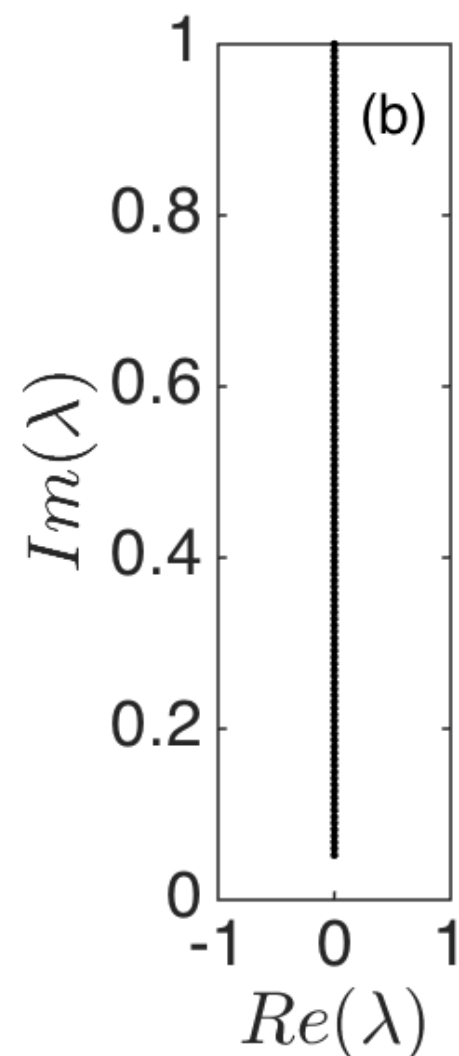
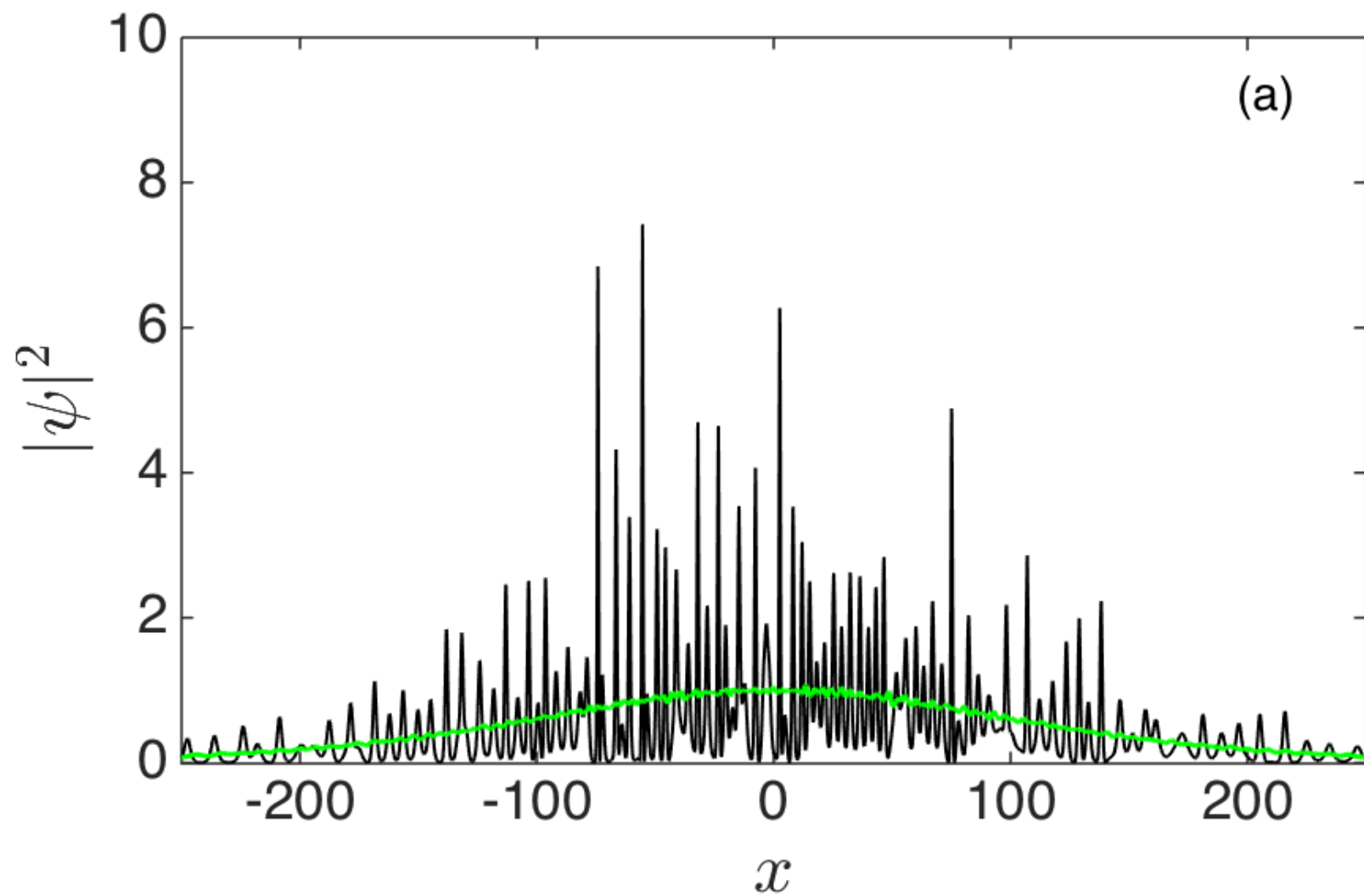
$$\lambda_n = i\eta_n = i \sqrt{1 - \left[\frac{\pi(n - 1/2)}{L} \right]^2}, \quad n = 1, 2, \dots, N, \quad N = \text{int}[L/\pi]$$



Black lines: 128-soliton solution with random phases, $x_{0,k} \sim \text{Random}[-2,2]$
Green lines: ensemble and time averaged first order moment

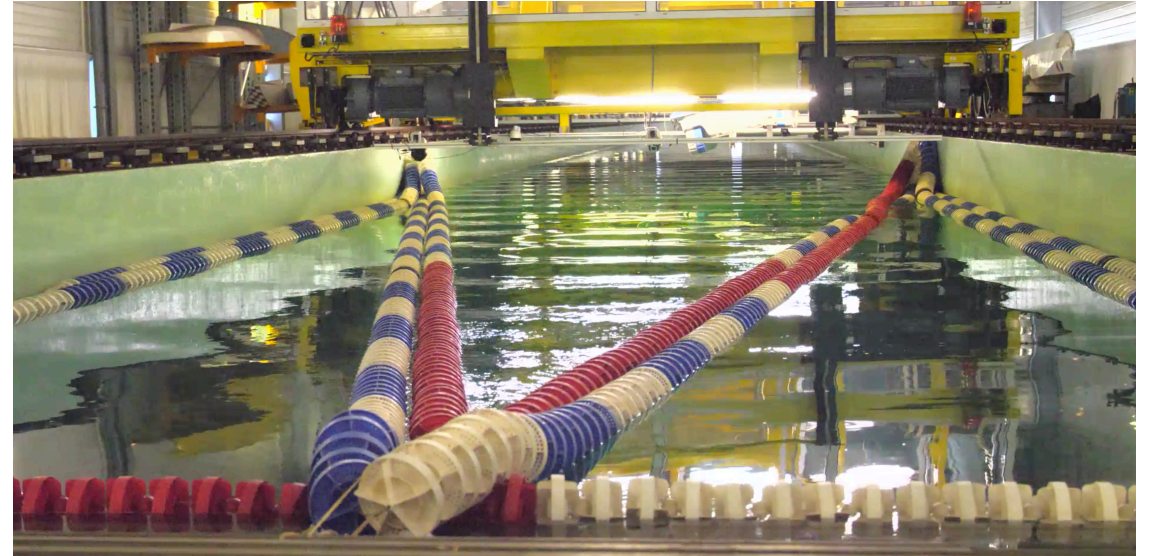


Blue lines: 64-soliton solution with random phases, $x_{0,k} = 0$
Red lines: the corresponding box wave field



Black lines: 128-soliton solution with random phases, $x_{0,k} \sim \text{Random}[-2,2]$
Green lines: ensemble and time averaged first order moment

Water wave tank in Nantes, France. 140 meters, 20 gauges



Integrable dynamics

$$\frac{\partial A}{\partial Z} = i \frac{k_0}{\omega_0^2} \frac{\partial^2 A}{\partial T^2} + i \alpha k_0^3 |A|^2 A$$

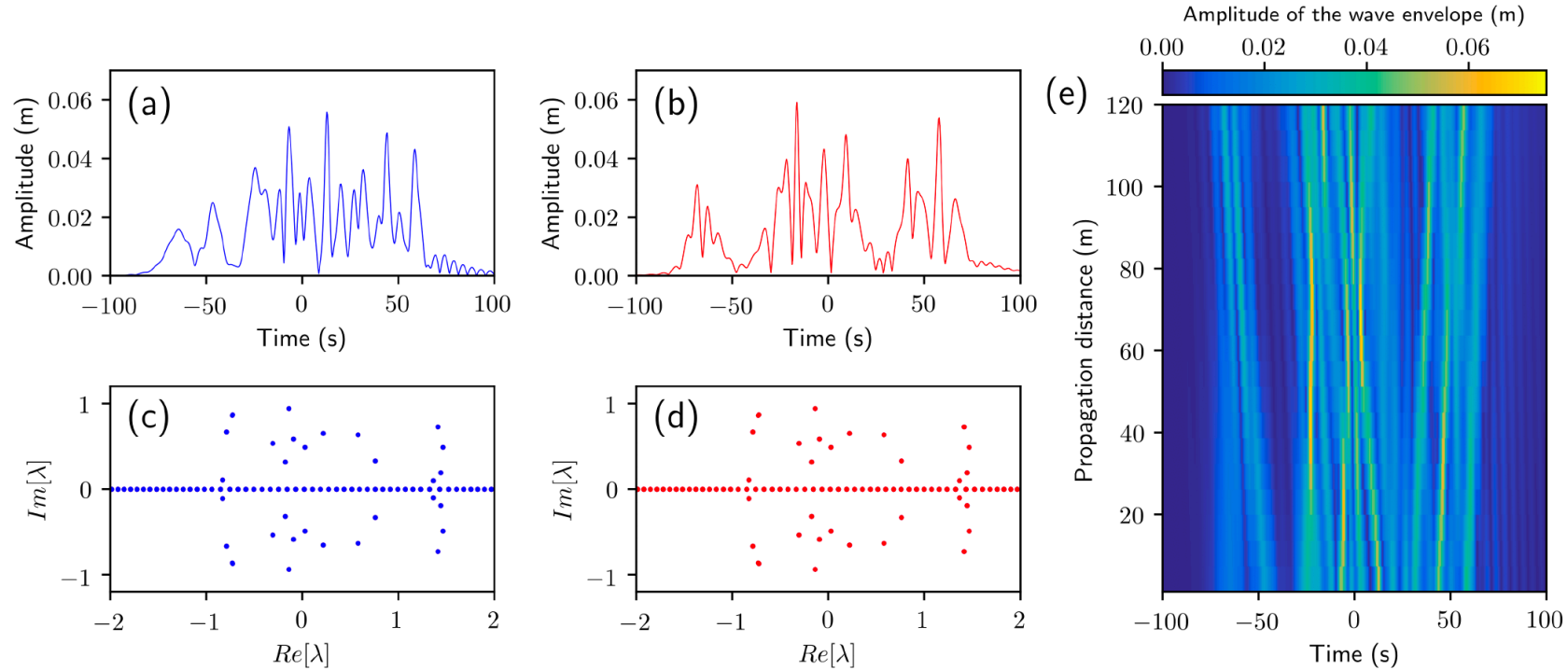


FIG. S1: Integrable dynamics. Numerical simulations of the integrable focusing 1D-NLSE (Eq. (S1) where the last three terms are neglected) for the ensemble of 16 solitons considered in Fig. 1 of the Letter. (a) Modulus $|A(Z_1, T)|$ of the wave envelope at $Z_1 = 6$ m and (c) corresponding discrete IST spectrum (red points). (b) Modulus $|A(Z_{20}, t)|$ of the wave envelope at $Z_{20} = 120$ m and (d) corresponding discrete IST spectrum. (e) Space-time plot showing the nonlinear evolution of the modulus $|A(Z, T)|$ of the wave envelope.

Experimental data

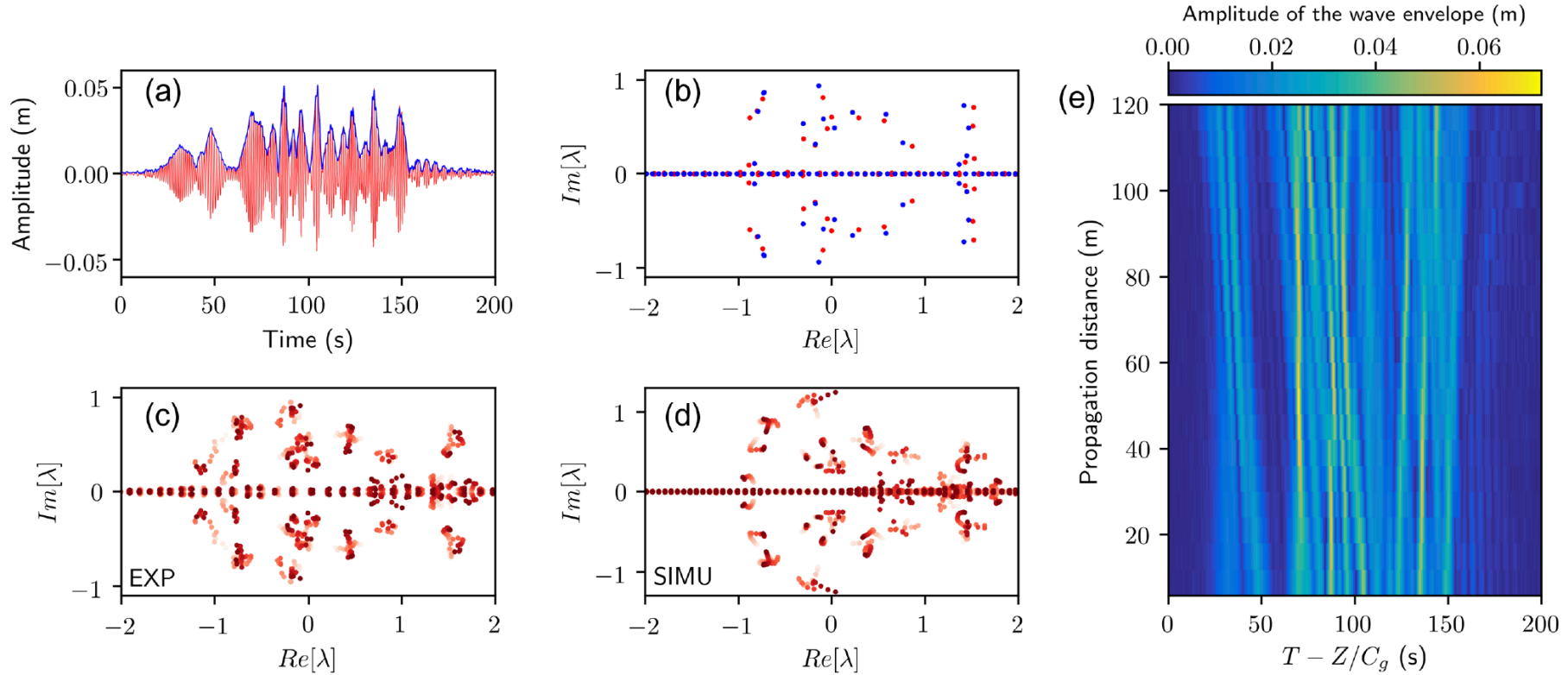


FIG. 1. Ensemble of $N = 16$ solitons propagating in the 1D water tank. (a) Water elevation (red line) and modulus of the wave envelope measured at $Z_1 = 6$ m, close to the wave maker. (b) Blue points represent the discrete IST spectrum of the numerically generated N-SS $\psi_{16}(x, t = 0)$ and red points represent the discrete IST spectrum measured at $Z_1 = 6$ m by using the signal plotted in (a). (c) Space evolution of the discrete IST spectra measured along the tank from $Z_1 = 6$ (light red) to $Z_{20} = 120$ m (dark red). (d) Same as in (c) but obtained from numerical simulations of a modified (not integrable) 1D-NLSE including higher-order effects; see Supplemental Material [53]. (e) Space-time evolution of modulus of the wave envelope recorded by the 20 gauges regularly spaced along the tank. Physical parameters characterizing the experiment are $f_0 = 0.9$ Hz, $k_0 = 3.26$ m $^{-1}$, $\alpha = 0.895$, $L_{NL} = 210$ m ($\langle |A_0(T)|^2 \rangle = 1.53 \times 10^{-4}$ m 2).

Experimental data

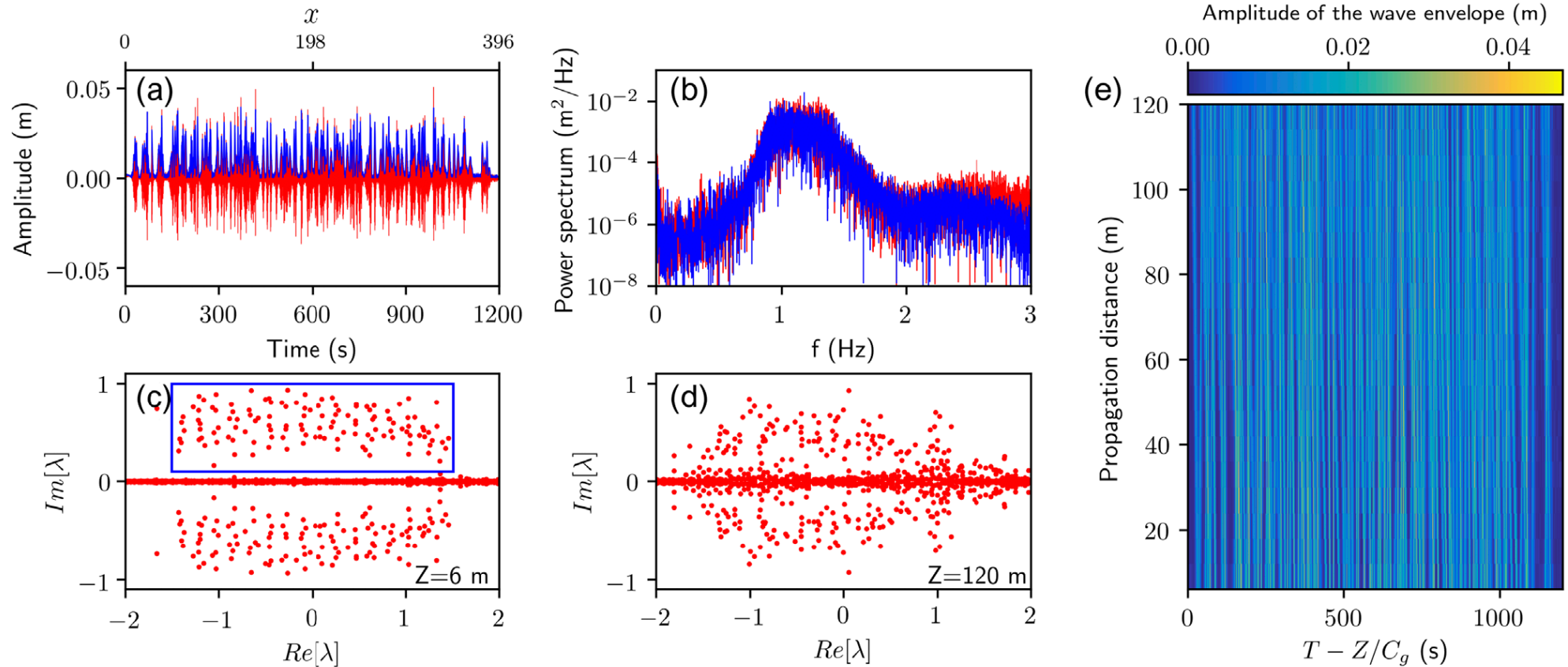


FIG. 2. Gas of $N = 128$ solitons propagating in the 1D water tank. (a) Water elevation (red line) and modulus of the wave envelope measured at $Z_1 = 6$ m, close to the wave maker. (b) Fourier power spectra of wave elevation at $Z_1 = 6$ (blue line) and at $Z_{20} = 120$ m (red line). (c) Discrete IST spectrum measured at $Z_1 = 6$ m. (d) Discrete IST spectrum measured at $Z_{20} = 120$ m. (e) Space-time evolution of modulus of the wave envelope recorded by the 20 gauges regularly spaced along the tank. Physical parameters characterizing the experiment are $f_0 = 1.15$ Hz, $k_0 = 5.32$ m^{-1} , $\alpha = 0.936$, $L_{\text{NL}} = 45$ m ($\langle |A_0(T)|^2 \rangle = 1.58 \times 10^{-4}$ m^2).

Publications:

- [1] A.A. Gelash, and D.S. Agafontsev, Strongly interacting soliton gas and formation of rogue waves, Phys. Rev. E., 2018.
- [2] A. Gelash, D. Agafontsev, V. Zakharov, G. El, S. Randoux and P. Suret, Bound state soliton gas dynamics underlying the noise-induced modulational instability, Phys. Rev. Lett., 2019.
- [3] P. Suret, A. Tikan, F. Bonnefoy, F. Copie, G. Ducrozet, A. Gelash, G. Prabhudesai, G. Michel, A. Cazaubiel, E. Falcon, G. El, S. Randoux, Nonlinear spectral synthesis of soliton gas in deep-water surface gravity waves. Phys. Rev. Lett., 2020, accepted.