Turbulence with Pressure

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Abstract

For one-dimensional compressible Euler's equations instead of the well-known Riemann's implicit solution an exact explicit analytical solution is obtained. To Riemann's shock wave arising time an explicit function on arbitrary initial conditions is also obtained. When the dissipation is taken into account, the possibility of solution retains smoothness for an unlimited time interval, is shown. An explicit form of solution gives possibility for deducing of the exact analytical representation also for anyone's one-point and multi-points momentums and spectrums of all hydrodynamics fields important to the turbulence theory. For example, explicit exact analytical solutions are obtained for second -, third -, and fourth-order structural functions, as well as for the energy spectrum of a turbulent flow. It is shown that near the collapse of a solution that does not take into account dissipation, the ratio between single-point moments of different orders for the density gradient indicates the effect of strong intermittency of turbulence.

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Introduction

Turbulence is an old and tantalizing subject when enormous amounts of data and ideas have been accumulated but still the problem is not solved [1]. The reason lies in the fact that the necessary exact solution [2-6] of the nonlinear hydrodynamics equations or field-theoretic tools [1] have appeared only in the last decades.

According to [7], the main problem of the theory of turbulence in a compressible medium is to obtain a closed description of any single-point or multi-point correlation (and cross - correlation) moments of density, pressure, and velocity field and the corresponding spectra. In this case, the specified hydrodynamic fields must be solutions of the corresponding Euler equations in the case of an ideal fluid or the Navier-Stokes equations for a compressible medium in the case of viscous dissipation.

In this paper, based on the approach developed in [4-6], we obtain an explicit analytical solution of the one-dimensional Euler equations for a compressible medium, which, like the Riemann solution, relates to the description of a nonlinear simple wave. In this case, the role of pressure is just accurately taken into account and its influence on the dynamics is decisive, in contrast to the theory of turbulence without pressure [1].

As a result, in this paper we obtain an exact explicit solution of the main problem of turbulence theory for a compressible medium in one-dimensional case.

1. Euler's equations

The one-dimensional Euler equation and continuity equation are represented in the form [8]:

$$\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0 \tag{1.1}$$

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho V)}{\partial x}=0 \tag{1.2}$$

For the polytropic gas, additionally to (1.1), (1.2), also the following relation is considered:

$$p=p_0(\frac{\rho}{\rho_0})^{\gamma} \tag{1.3}$$

Assuming for the case of simple wave that V can be also represented as a function of ρ only, following relations are obtained in [8] (see (101.4) in [8]):

$$\frac{dV}{d\rho}=\pm\frac{c}{\rho}; \qquad c^2=\frac{dp}{d\rho} \tag{1.4}$$

In the result, from (1.1), (1.2) and (1.4), the following modification of the one-dimensional hydrodynamic equations can be inferred [8, 9]:

$$\frac{\partial V}{\partial t}+(V\pm c)\frac{\partial V}{\partial x}=0 \tag{1.5}$$

$$\frac{\partial \rho}{\partial t}+(V\pm c)\frac{\partial \rho}{\partial x}=0 \tag{1.6}$$

$$\frac{\partial c}{\partial t}+(V\pm c)\frac{\partial c}{\partial x}=0 \tag{1.7}$$

$$\frac{\partial p}{\partial t}+(V\pm c)\frac{\partial p}{\partial x}=0 \tag{1.8}$$

From (1.5) and (1.7), in particular, after adding and subtracting one equation from another, the Riemann equation is represented in the next form, known also as the Hopf-Burgers (HB) equation:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0; \qquad u=V\pm c \tag{1.9}$$

2. Explicit solution of the Riemann problem

Let us get from (1.5)-(1.9) an explicit form for the velocity field using approach developed in [2-6].

$$V(x,t)=\int_{-\infty}^{\infty}d\xi V_0(\xi)(1+t\frac{du_0(\xi)}{d\xi})\delta(\xi-x+tu_0(\xi)), \qquad u_0(\xi)=V_0(\xi)\pm c_0(\xi); \tag{2.1}$$

Thus, representation (2.1) gives generalization of known solutions [2-6] to the case of one-dimension Euler equation with nonzero pressure gradient when $c_0 \neq 0$ in (2.3) and so in (2.3) is the explicit analytical form of the Riemann solution for the simple wave.

For the distributed density, we also have the following exact solutions of the equations (1.6) [4-6]:

$$\rho(x,t) = \int_{-\infty}^{\infty} d\xi \rho_0(\xi)(1+t\frac{du_0}{d\xi})\delta(\xi - x + tu_0(\xi)); \rho_0(x) = \rho(x,t=0) \qquad \textbf{(2.2)}$$

Consider, for example, dynamics of the density partial derivative over space variable:

$$\frac{\partial\rho(x,t)}{\partial x} = \int_{-\infty}^{\infty} d\xi \frac{d\rho_0(\xi)}{d\xi}\delta(\xi - x + tu_0(\xi)) \qquad \textbf{(2.3)}$$

From (2.3) for the one-point moment of second order in turbulence theory, we get close-form representation:

$$(\frac{\partial\rho}{\partial x})^2 = \int_{-\infty}^{\infty} d\xi \frac{(d\rho_0 / d\xi)^2 \delta(\xi - x + tu_0(\xi))}{1 + tdu_0 / d\xi} \qquad \textbf{(2.4)}$$

Actually, the solutions loose smoothness at some time instance $t = t_0$, depending on the initial velocity field distribution. Value t_0 is defined as the minimum time such that the following equality holds (see also [4-6]):

$$1 + t(\frac{dV_0(x)}{dx} \pm \frac{dc_0(x)}{dx}) = 0;$$
$$t_0 = \frac{1}{\max\left|\frac{du_0(x)}{dx}\right|}; u_0 \equiv V_0 \pm c_0 \qquad \textbf{(2.5)}$$

For example, from (3.1) for the initial velocity distributions $V_0(x) = a\exp(-x^2 / 2x_0^2)$ and corresponding initial fields of density and local speed of sound (2.15) and (2.16) the minimal time of the shock wave arising is:

$$t_0 = \frac{2x_0\sqrt{e}}{a(\gamma + 1)} \qquad \textbf{(2.6)}$$

3. Solution regularization by dissipation

It is known [10] that the presence of viscosity and thermal conductivity can lead to the dissipation of the wave energy. Here the increment $\mu = const$ characterizes the value of dissipative factors in the absence of dispersion (see (81.10) in [2]). In this case, for example, equation (1.9) has the form:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -u\mu \qquad \textbf{(3.1)}$$

In all other equations (1.5)-(1.8) it is also need to enter in the right hand side the same linear term for the corresponding field.

When such dissipative factors are taken into account above, the expression (2.4) has the following representation:

$$(\frac{\partial\rho}{\partial x})^2 = e^{-2\mu t}\int_{-\infty}^{\infty} d\xi \frac{(d\rho_0/d\xi)^2\delta(\xi - x + \tau(t)u_0(\xi))}{1+\tau(t)du_0/d\xi};$$
$$\tau(t) = \frac{1-\exp(-t\mu)}{\mu} \qquad \textbf{(3.2)}$$

4. Turbulence theory

In the theory of turbulence, one of the main characteristics of a turbulent flow is the structural function of the hydrodynamic field under study [7]. The Kolmogorov's law 4/5 gives for a third-order longitudinal structural function exact representation $S_3^l(r) = -\frac{4}{5}\langle\varepsilon\rangle r$.

Based on the exact solution (2.1) and taking into account that $u_0(x) = c_\infty + \frac{(\gamma+1)}{2}V_0(x)$, we consider the structural function of the velocity field, which we define as:

$$S_p = \frac{1}{L}\int_{-\infty}^{\infty} dx[V(x+r;t) - V(x;t)]^p \qquad \textbf{(4.1)}$$

In (4.1) $p = 2;3;4;..$ and introduced a characteristic integral length scale [5, 6]:

$$L = \frac{P_0^2}{2\sqrt{\pi}E_0}; P_0 = \int_{-\infty}^{\infty} dx|V_0(x)|; E_0 = \int_{-\infty}^{\infty} dx V_0^2(x); V_0(x) = V(x;t=0) \qquad \textbf{(4.2)}$$

For example, it is possible to obtain $S_2 = \frac{2}{L}\sum_{n=1}^{\infty}\frac{r^{2n}(-1)^{n+1}}{(2n)!}\int_{-\infty}^{\infty} dx\left(\frac{d^nV_0}{dx^n}\right)^2$. At the initial velocity field $V_0(x) = a\exp(-x^2/2x_0^2)$ characteristic length scale $L = x_0$ and we get that $S_2 = r^2a^2\sqrt{\pi}/2x_0^2 - O(r^4/x_0^4)$ and also that for $p = 4$ in (4.1) we obtain $S_4 \approx \frac{9r^4a^4}{4x_0^4}\sqrt{\frac{\pi}{2}}$. Thus, non-Gaussian relation $\frac{S_4}{3S_2^2} = \frac{3}{\sqrt{2\pi}} \approx 1.196 > 1$ is valid.

Similarly, for the exponent value $p = 3$ in (5.1), we obtain the representation for a third-order structural function:

$$S_3 = \frac{3(\gamma+1)t}{2L}\sum_{n=1}^{\infty}(-1)^n\frac{r^{2n-1}}{(2n-1)!}\int_{-\infty}^{\infty} dx\left(\frac{d^n}{dx^n}V_0^2\right)^2 \qquad \textbf{(4.3)}$$

We now consider the energy spectrum and to determine it, we introduce a correlation function, which for the considered exact solution of the Euler equations (2.3) has the form:

$$R(r) = \frac{1}{L}\int_{-\infty}^{\infty} dx V(x+r;t)V(x,t) = \frac{1}{L}\int_{-\infty}^{\infty} d\xi_1\int_{-\infty}^{\infty} d\xi_2 V_0(\xi_1)A(\xi_1)V_0(\xi_2)A(\xi_2)B(\xi_1;\xi_2;r); \qquad \textbf{(4.4)}$$

$$A(x) = 1 + \frac{(\gamma+1)t}{2}\frac{dV_0}{dx}; B = \delta(\xi_2 - \xi_1 - r + \frac{(\gamma+1)t}{2}(V_0(\xi_2) - V_0(\xi_1))) \qquad \textbf{(4.5)}$$

The Fourier transform of the correlation function (4.5) allows us to obtain the following exact closed representation for the energy spectrum, expressed in terms of an arbitrary initial velocity field in the form:

$$E(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} drR(r)\exp(ikr) = \frac{1}{2\pi L} I(k)I(-k);$$

$$\text{where } I(k) = \int_{-\infty}^{\infty} dxV_0(x)A(x,t)\exp\left(ik\left(x + t\frac{(\gamma+1)}{2}V_0(x)\right)\right) \text{ and } e = \int_0^{\infty} dkE(k) \qquad \textbf{(4.6)}$$

As a result, in the limit $kL >> 1$ and $t \to t_0$ for the energy spectrum (4.6) with an arbitrary initial velocity field, we obtain a universal dependence on the wave number in the form of the law-8/3:

$$E(k) = C_E k^{-8/3}$$

$$\text{where } C_E = \frac{2^{5/3}}{L}\left(\frac{dV_0}{dx}\right)^{8/3}_{x=x_M}\left(\frac{d^3V}{dx^3}\right)^{-2/3}_{x=x_M}\Phi^2(0) \qquad \textbf{(4.7)}$$

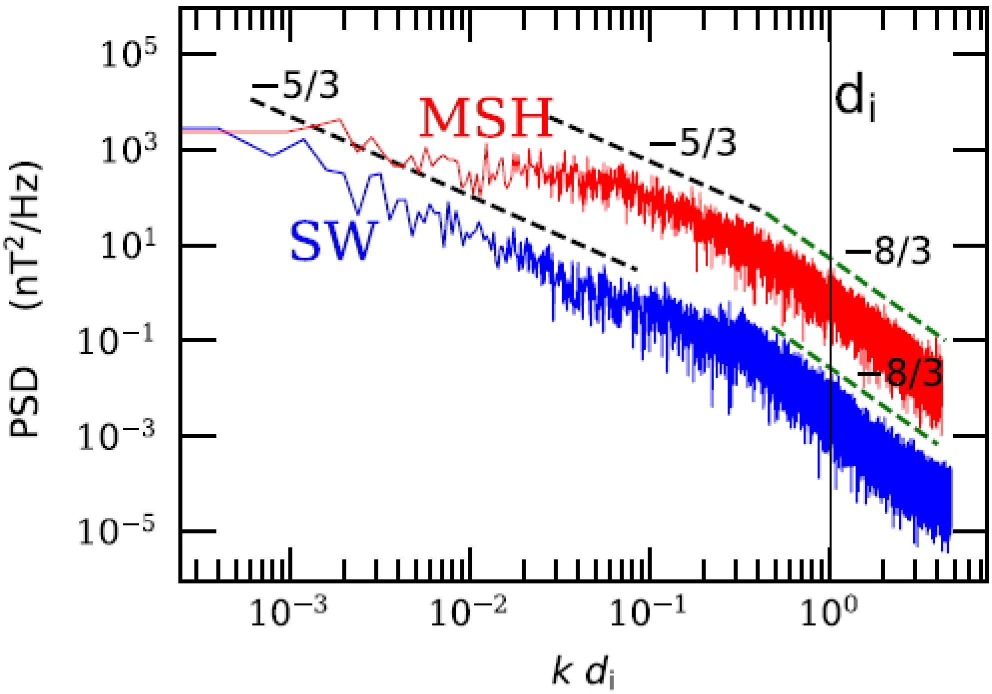
In (4.7) $\Phi(0) = \frac{1}{\sqrt{\pi}}\int_0^{\infty} du\cos(\frac{u^3}{3}); \Phi(z) = \sqrt{\pi}Ai(z)$ - the Airy function and [11] (see formula (b. 8) on page 784 in [11] where $\Phi(0) = \frac{\sqrt{\pi}}{3^{2/3}\Gamma(2/3)} \approx 0.629$).

For example, in the case of initial velocity field $V_0 = a\exp(-x^2/2x_0^2)$ from (5.10) it is possible to obtain $E(k) = \frac{a^2x_0\Phi^2(0)}{(kx_0)^{8/3}2^{-1/3}e\sqrt{\pi}}$.

The spectrum of density pulsations will have a similar power-law dependence on the wave number. In this regard, it is of interest to compare the turbulent spectrum (4.7) with the results of observations of turbulent spectra of electron density pulsations and correlated with it magnetic field pulsations in the magnetosheath [12], where an exponent close to the power-law dependence of the turbulent spectrum (4.7) on the wave number is obtained. Similar values of exponents close to -8/3 (see Fig. 1 below taking from [12]) were obtained in the observation data of turbulent spectra for the Solar Wind and for the Earth's magnetosheath.

The obtained correspondence of the spectrum (4.7) with the specified data may indicate an additional mechanism of the sink to the turbulent energy, which is not associated with ordinary dissipation, but is caused by the process of nonlinear overturning of a simple wave. At the same time, the effects of compressibility are significant, especially in the observations of turbulent spectra in the magnetosheath. Moreover, according to [12] , for the magnetosheath region of the gentle slope $\propto k^{-1}$ of the spectrum (the region of energy pumping) with an increase in the wave number directly passes into the region with a steep slope $k^{-8/3}$ (see Fig.1 below) really bypassing the intermediate regime with the Kolmogorov-Obukhov spectrum $\propto k^{-5/3}$.

Fig.1



Magnetic field turbulence spectra for the solar-wind (SW) in blue (when $d_i = 75\,\text{km}$; $\left|\langle \vec{V} \rangle\right| = 330\,\text{km s}^{-1}$) and magnetosheath (MSH) in red (when $d_i = 56\,\text{km}$; $\left|\langle \vec{V} \rangle\right| = 278\,\text{km s}^{-1}$) interval. The solid vertical line represents $kd_i = 1$ with the wave vector $k = 2\pi f / \left|\langle \vec{V} \rangle\right|$, where f is the frequency [12] (see Fig.1 in [12]).

Let's consider the single-point moments of hydrodynamic fields. Examples of a single-point moment for a density gradient without and with dissipation have already been given in (2.4) and (3.2). We obtain that the averaged single-point moment of the density gradient of any order n has the form:

$$G_n \equiv \frac{1}{L}\int_{-\infty}^{\infty} dx \left(\frac{d\rho(x,t)}{dx}\right)^n = \frac{1}{L}\int_{-\infty}^{\infty} dx \left(\frac{d\rho_0(x)}{dx}\right)^n \frac{1}{\left(1 + t\dfrac{du_0}{dx}\right)^{n-1}} \qquad \textbf{(4.8)}$$

In (4.8), for example, for a polytropic medium with an initial velocity field $V_0(x) = a\exp(-x^2/2x_0^2)$, representations $\rho_0(x) = \rho_\infty \left(1 \pm \frac{(\gamma-1)a}{2c_\infty}\exp(-x^2/2x_0^2)\right)^{\frac{2}{(\gamma-1)}}$; $\frac{du_0}{dx} = \frac{(\gamma+1)ax}{2x_0^2}\exp(-x^2/2x_0^2)$ should be used. Thus without taking into account the dissipation in (4.8) torque density gradient has a singularity $G_n \propto O\left(1/(1-t/t_0)^{n-1}\right)$ in the limit $t \to t_0$ where the minimum time of existence of smooth solutions given in (3.3) for considered initial velocity field. In this limit, there is a relation $G_{2n}/G_n^2 \propto 1/(1-t/t_0) >> 1$. That is typical for the regime of strong intermittency of turbulence [7, 8]. Indeed, the representation for the moment of the density gradient is obtained from (4.8) when $n \to 2n$ replaced in (4.8).

Conclusions

Thus the exact closed-form explicit analytical solution to the Riemann problem for the Euler one-dimensional hydrodynamics equations is obtained.

The regularization by dissipation factors is determined for that solution for unlimited time, which gives unexpected positive resolution for the generalization of the Clay problem (www.claymath.org) to the fluid and gas dynamics in the compressible case.

On the base of explicit analytical form of the Riemann solution the explicit representation for the shock wave arising time is obtained for arbitrary initial conditions of the simple wave.

Closed explicit analytical representations are obtained for one-point and two-point moments of hydrodynamic fields and for the energy spectrum, which gives an example of solving the turbulence problem based on the exact solution of one-dimensional Euler equations for a compressible medium.

In this case, the turbulence spectrum power-law obtained from the exact solution of the Euler equation corresponds to the parameters of the turbulent spectra observed in the Earth's and Saturn's magnetospheres, as well as for the Solar Wind [12].

Based on this correspondence, we propose a mechanism for the sink of the turbulent energy, which was not considered earlier and which is caused by the collapse of nonlinear simple waves described by an explicit solution of the Euler equation. Understanding how collisionless plasmas dissipate remains a topic of central importance in space physics, astrophysics, and laboratory plasma. In recent years, it has become increasingly recognized that the MHD description must be refined to clearly make a connection with kinetic plasma dissipation [12]. The present results provide a step toward understanding this problem on the basis of the exact solution for compressible Euler's equations.

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