Point vortices dynamics on a rotating sphere and modeling of global atmospheric vortices interaction (Physics of Fluids 32, 106605 (2020); https://doi.org/10.1063/5.0026014)

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Abstract

It is shown that the hydrodynamics equations for a thin spherical liquid layer are satisfied by the stream function of a pair of antipodal vortices (APV), in contrast to the stream function of a single point vortex on a sphere with a background of a uniform opposite sign vorticity. A simple zero solution of the equation of the absolute vorticity conservation is used for bypassing well-known nonlinear problem of a point vortices interaction with regular vorticity field and an exact solution for APVs dynamics problem on a rotating sphere is obtained. Due to this a new stable stationary solution for the dynamics of APV is obtained, which can model the dynamics of the global vortex structures such as atmospheric centers of action.

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1. Introduction

The key large-scale vortex structures in the atmosphere are atmospheric centers of action (ACA). Regional and large-scale atmospheric anomalies and their changes are related to the ACAs regimes. For example, atmospheric conditions in the Northern Hemisphere over Europe are related to quasi-steady Icelandic cyclone (IC) and Azores anticyclone (AA) over North Atlantic.

For modeling of dynamics of vortices like ACA dynamics the system of interacting antipodal point vortices (APV) is used [1]-[4]. Each APV is a combination of two point vortices located at diametrically conjugated points of the sphere and having equal magnitude, but opposite sign circulation.

Up to now absolute stationary solutions to the dynamics equations of APVs on a rotating sphere, which could simulate the stationary or quasi-stationary states like for the centers of action in the atmosphere are not known [5]-[7].

In this paper, a solution is obtained for the dynamics of APV on a rotating sphere, for which there is a stable stationary mode that simulates exactly the absolute stationary ACA-like structure corresponding to the condition $\omega = 0$.

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2. Revision of the problem of elementary singular vortex object on a sphere

In the spherical coordinate system (r,θ,φ), rigidly connected to the globe and with the origin in the center of the globe the hydrodynamics equations with zero radial component of velocity $V_r=0$ for the stationary and azimuthally symmetrical case are represented in the most simple form, when $V_\theta=\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}=0$; $V_\varphi=-\frac{1}{r}\frac{\partial\psi}{\partial\theta}\neq 0$:

$$\frac{(V_\varphi+\Omega r\sin\theta)^2}{r}=\frac{1}{\rho_0}\frac{\partial p}{\partial r};\qquad (1)$$

$$\frac{(V_\varphi+\Omega r\sin\theta)^2\cot\theta}{r}=\frac{1}{\rho_0 r}\frac{\partial p}{\partial\theta}$$

In (1) $\rho_0=const$ is the density, $\Omega=const$ is the constant speed of rotation of the globe or sphere, and p is the pressure.

A unit point vortex on a sphere together with compensating uniform regular vorticity field is considered by, e.g. Bogomolov [3], Kimura and Okamoto [8]; Dritschel and Boatto [9]-[11] gives stream function:

$$\psi = \frac{\Gamma_1}{2\pi}\ln\left(\frac{1+\cos\theta}{1-\cos\theta}\right) \qquad (2)$$

Stream function (2) does not meet (1) when rotation is absent for $\Omega = 0$ (see also [12]), whereas the stream function of APV satisfies them in this case. Stream function of APV has representation:

$$\psi_1 = \frac{\Gamma_1}{4\pi}\ln\left(\frac{1}{1-\cos\theta}\right) \qquad (3)$$

.

In the case when rotation of a sphere with angular velocity Ω is taken into account in (1), one needs to replace , $V_\varphi \to V_\varphi + \Omega r \sin\theta$ and, accordingly, use the stream functions (2) and (3) with an additional term $\Psi_0 = -\Omega r^2 \cos\theta$.

$$\psi = -\Omega r^2 \cos\theta + \frac{\Gamma_1}{2\pi}\ln\frac{1+\cos\theta}{1-\cos\theta} \qquad (4)$$

3. APV dynamics on a rotating sphere

The dynamic interaction of APVs is considered based on an exact weak solution of the absolute vorticity conservation equation on a rotating sphere which for the case of constant thin of spherical layer is [6], [13]:

$$\frac{\partial\omega}{\partial t}+\frac{V_\theta}{R}\frac{\partial\omega}{\partial\theta}+\frac{V_\varphi}{R\sin\theta}\frac{\partial\omega}{\partial\varphi}=0 \qquad (5)$$

where $\omega=\omega_r+2\Omega\cos\theta$ is the angular velocity of the sphere rotation (for the Earth, $\Omega\approx 7.3\times10^{-5}\,\mathrm{sec}^{-1}$), θ is the co-latitude; φ is the longitude;

$V_\theta=R\frac{d\theta}{dt}, V_\varphi=R\sin\theta\frac{d\varphi}{dt};\omega_r=\frac{1}{R\sin\theta}\left(\frac{\partial V_\varphi\sin\theta}{\partial\theta}-\frac{\partial V_\theta}{\partial\varphi}\right)=-\Delta\psi$ is the radial component of the local vortex field on the sphere, is Δ s the Beltrami- Laplace operator; is the stream function. According to (5), each Lagrangian particle preserves the value of absolute vorticity in a thin layer of liquid on a rotating sphere. Therefore, as in [14]-[16], we use the absolute vortex field in the form of APVs system:

$$\omega=-\Delta\psi+2\Omega\cos\theta=\hat{L}(\delta);$$

$$\hat{L}(\delta)=\frac{1}{R^2}\sum_{i=1}^{N}\frac{\Gamma_i}{\sin\theta_i}\left(\delta(\theta-\theta_i)\delta(\varphi-\varphi_i)-\delta(\theta+\theta_i-\pi)\delta(\varphi-\varphi_i-\pi)\right) \qquad (6)$$

As a result, we obtain an exact solution of equation (6) for the stream function in the form:

$$\psi=-\Omega R^2\cos\theta+\frac{1}{2\pi}\sum_{i=1}^{N}\Gamma_i\ln\frac{1+\cos u_i}{1-\cos u_i} \qquad \cos u_i=\cos\theta\cos\theta_i+\sin\theta\sin\theta_i\cos(\varphi-\varphi_i);\Gamma_i=const \qquad (7)$$

Note that in the case of $N=1$ the stream function (7) exactly coincides with the stream function (4) for a single APV located at the poles of the sphere.
A weak solution to (5) is obtained by substituting in (5) the vortex field (6) and corresponding steam function (7). As a result, the functions $\theta_i(t),\ \varphi_i(t)$ are defined as the solutions to the following *2N*-dimensional Hamiltonian system of ordinary differential equations [14]-[16]:

$$\frac{d\theta_i}{dt} = \dot{\theta}_i = -\frac{1}{\pi R^2}\sum_{\substack{k=1\\k\neq i}}^{N}\frac{\Gamma_k sin\theta_k \sin(\varphi_i-\varphi_k)}{1-cos^2 u_{ik}}$$

$$cosu_{ik} = cos\theta_i cos\theta_k + sin\theta_i sin\theta_k \cos((\varphi_i-\varphi_k)$$

$$\frac{d\varphi_i}{dt} = \dot{\varphi}_i = -\Omega - \frac{1}{\pi R^2}\sum_{\substack{k=1\\k\neq i}}^{N}\frac{\Gamma_k(cot\theta_i sin\theta_k \cos(\varphi_i-\varphi_k)-cos\theta_k)}{1-cos^2 u_{ik}} \qquad (8)$$

The system (8) for with accuracy up to a multiplier (set to п) coincides with the corresponding system derived in [4] for N pairs of APVs.
The system (8) for $\Omega=0$ with accuracy up to a multiplier (set to п) coincides with the corresponding system derived in [4] for N pairs of APVs.

4. Steady vortex modes (N=2) and their stability

Let us consider steady modes corresponding to the equilibrium for $N=2$ APVs in (8).

$$\frac{\Gamma_1}{\Gamma_2} = -\frac{\sin\theta_{20}}{\sin\theta_{10}} \qquad (9)$$

An additional condition for the absence of absolute motion uses the equality $\frac{d(\varphi_1+\varphi_2)}{dt}=0$, and has the form:

$$\Gamma_2 = -\Omega\pi R^2 \sin\theta_{10}\sin(\theta_{20}-\theta_{10}) < 0; \Omega > 0, \theta_{20} > \theta_{10} \qquad (10)$$

From (9), it follows that $\gamma_1 = \Gamma_1/\Gamma_2 < 0$. It means that we consider a steady mode for two APVs having opposite circulation directions and placed on the same meridian. Also, from (9), it follows that $|\Gamma_2| < |\Gamma_1|$ and, thus, APV with $\theta=\theta_{20}$ has less intensity than APV with $\theta=\theta_{01}$, if, by definition, we assume $\theta_{20}>\theta_{10}$ in (9) and (10). Moreover from (10) we can determine that APV with the coordinate $\boldsymbol{\theta_{20}}$ and intensity $\boldsymbol{\Gamma_2}$ is indeed anticyclone because an anticyclone having the direction of rotation opposite to the direction of the rotation of the sphere.

Indeed, for the ACA over the North Atlantic, the Meridian coordinates of the Icelandic cyclone and the Azores anticyclone are close each to other. This corresponds to the case $\varphi_1 = \varphi_2 = const$ under which the conditions (9) and (10) are obtained. According to (9), ACA over North Atlantic have opposite signs and different values of the vortex circulation. As in (9), in reality, the intensity of the Icelandic cyclone usually exceeds the intensity of the Azores anticyclone.

Consider now stability of the steady mode (9), (10), modeling steady regime of a cyclonic-anti-cyclonic ACA pair. We use for this the system (8) for the case N=2.

Let us introduce a disturbance of the steady mode (9), (10) as

$$x = \theta_1(t) - \theta_{10}; z = \theta_2(t) - \theta_{20}; y = \varphi_1(t) - \varphi_2(t) \qquad (11)$$

Then, the system of equations for the disturbances (11) is as follows

$$\begin{aligned} \frac{dx}{d\tau} &= \sin\theta_{10}\sin(\theta_{20}+z)\sin y, \\ \frac{dz}{d\tau} &= \sin\theta_{20}\sin(\theta_{10}+x)\sin y, \\ \frac{dy}{d\tau} &= -\sin\theta_{10}\cos(\theta_{20}+z) - \sin\theta_{20}\cos(\theta_{10}+x) + W\cos y, \end{aligned} \qquad (12)$$

$$\tau = t\Omega\frac{\sin(\theta_{20}-\theta_{10})}{1-U_0^2}; \qquad W = \sin\theta_{10}\cot(\theta_{10}+x)\sin(\theta_{20}+z) + \sin\theta_{20}\cot(\theta_{20}+z)\sin(\theta_{10}+x)$$

$$U_0 \equiv U(\tau=0) = \cos(\theta_{10}+x(0))\cos(\theta_{20}+z(0)) + \sin(\theta_{10}+x(0))\sin(\theta_{20}+z(0))\cos y(0)$$

The system (12) describes non-linear evolution of the disturbances. Despite the obtained stability of the dynamics of the perturbed vortex system, the relative magnitude of change of the vortices location during their oscillations near the stationary state substantially depends on the form of the initial perturbations and their amplitude (see Fig.1).

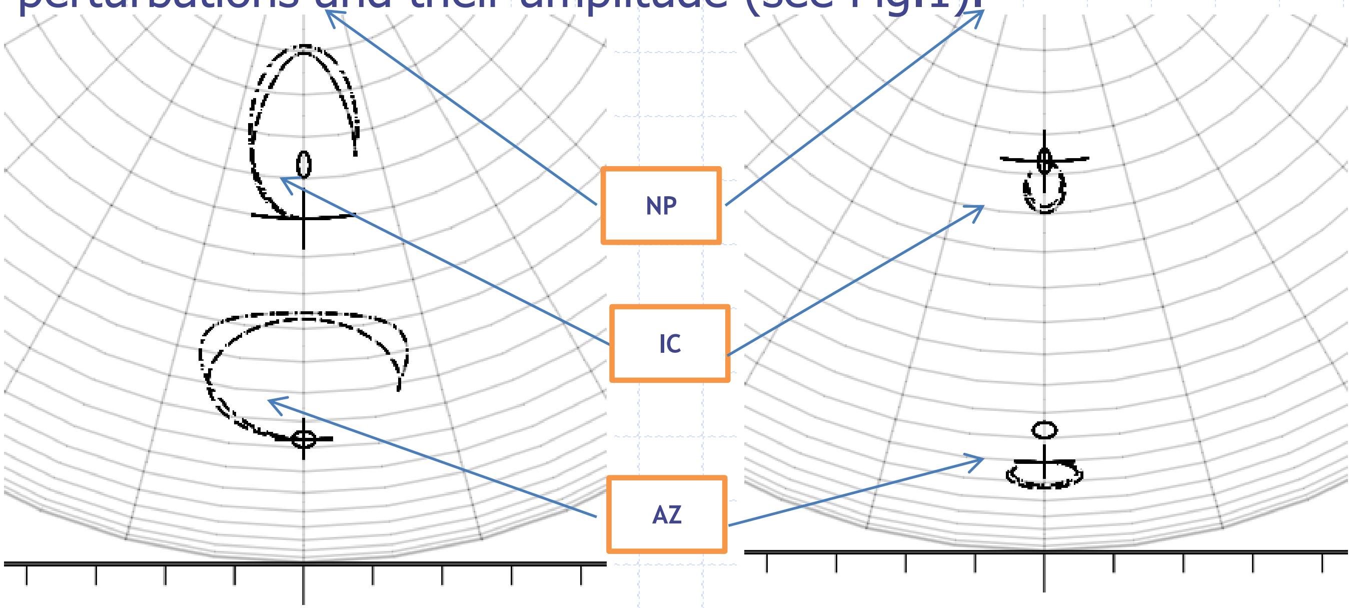

Fig. 1. Plots of periodic trajectories of Icelandic (IC, about equilibrium $\theta_{10} = 25^0$ shown by a small solid ellipse) and Azores (AZ, about equilibrium $\theta_{20} = 55^0$ shown by a small solid ellipse) vortices according to (12) and its linearization for various initial conditions (shown by crosses) in the Northern hemisphere. Simulation time is less than a period. a) x(0)=5^0, z(0)=y(0)=0; solutions of linearized of the (12), dash, mainly inner lines, and solutions of non-linear equations (12), dash-dot, mainly outer lines; b) z(0)=5^0, x(0)=y(0)=0, solutions of linear equations are outer lines, and solutions of (12) are inner lines The North Pole (NP) is the point of the meridians (radial lines with stride 15^0) intersection (not visible). Concentric lines represent latitudes with stride 3.75^0.

Thus, significant weather and climatic anomalies can be associated only with initial displacements of the Icelandic cyclone in the south direction (see Fig.1a).

According to a long-term observations (see Appendix C in [15]) the cyclonic-anticyclonic vortex pair of the ACAs over North Atlantic, usually only slightly deviates from the equilibrium annual-mean coordinates that corresponds to the case illustrated by Fig. 1b. However, for some years, significant deviations are observed as shown in Fig. 1a. The mechanism for the formation of such dynamics with significant difference in behavior, as manifested for atmospheric centers of action in Fig. 1a and Fig. 1b, requires clarification.

5. Polar vortex and stability observation data

Let us consider the case when a cyclonic anti-cyclonic pair of APV interacts with an APV located on the poles with the stream-function $\psi_P = \frac{\Gamma_0}{2\pi} \ln\left(\frac{1+\cos\theta}{1-\cos\theta}\right)$ to be added in the right-hand side of (7) [15], [16]. Then, it is necessary only make the following substitutions in (8):

$$\frac{d\varphi_i(t)}{dt} \to \frac{d\varphi_i(t)}{dt} + \frac{\Gamma_0}{\pi R^2 \sin\theta_i(t)} \qquad (13)$$

Taking into account (13), instead of equilibrium conditions (9) and (10), the new conditions obtained for the realization of the stationary state of the cyclonic anti-cyclonic pair of APV on the rotating sphere with a polar APV having non-zero circulation Γ_0:

$$\frac{\Gamma_0}{\Gamma_2} = \left(\frac{\Gamma_1}{\Gamma_2} + \frac{\sin\theta_{20}}{\sin\theta_{10}}\right) \frac{\sin^2\theta_{10}}{\left(\sin^2\theta_{20} - \sin^2\theta_{10}\right)\sin(\theta_{20}-\theta_{10})} \qquad (14)$$

$$\frac{\Gamma_0}{\Gamma_2} = \frac{\sin\theta_{10}\sin\theta_{20}}{\left(\sin^2\theta_{10} + \sin^2\theta_{20}\right)} \left[\frac{\sin\theta_{20}}{\sin(\theta_{20}-\theta_{10})} - \frac{\Gamma_1}{\Gamma_2}\frac{\sin\theta_{10}}{\sin(\theta_{20}-\theta_{10})} + \frac{2\pi R^2 \Omega \sin\theta_{10}\sin\theta_{20}}{\Gamma_2}\right] \qquad (15)$$

Stability conditions for the stationary state (14), (15) in the limit of extremely small disturbances are as follows [15], [16]:

$$D = A\gamma_1^2 + 2B\gamma_1 + C < 0; \gamma_1 = \frac{\Gamma_1}{\Gamma_2};$$
$$A = \sin^3\theta_{10}\left[\sin(2\theta_{20} - \theta_{10}) + \frac{2\sin^2\theta_{10}\cos\theta_{20}\sin(\theta_{20} - \theta_{10})}{\sin^2\theta_{20} - \sin^2\theta_{10}}\right];$$
$$B = \sin\theta_{10}\sin\theta_{20}\left(\sin^2\theta_{20} + \sin^2\theta_{10}\right);\quad (16)$$
$$C = \sin^3\theta_{20}\left[\sin(2\theta_{10} - \theta_{20}) + \frac{2\sin^2\theta_{20}\cos\theta_{10}\sin(\theta_{20} - \theta_{10})}{\sin^2\theta_{20} - \sin^2\theta_{10}}\right]$$

In particular, for the equilibrium coordinates of the vortex pair of APV $\theta_{20} = 55^0; \theta_{10} = 25^0$, the condition (16) yields the following inequality:

$$1.36 < \left|\frac{\Gamma_1}{\Gamma_2}\right| < 5.11 \quad (17)$$

Then the intensity of the polar vortex corresponding to the stationary state (14), (15) is $\Gamma_0 / \Gamma_2 = 0.725(1.94 - |\Gamma_1/\Gamma_2|); \Gamma_1 > 0; \Gamma_2 < 0$. For $|\Gamma_1/\Gamma_2| > 1.94$, it corresponds to the cyclonic circulation orientation of the polar vortex. Vice versa, anti-cyclonic circulation of the polar vortex corresponding to observations [15] satisfies $|\gamma_1| = |\Gamma_1/\Gamma_2| < \sin\theta_{20}/\sin\theta_{10}$ following from (14).

If the condition (16) is violated, i.e. when for $D > 0$, the stationary state (14), (15) is exponentially unstable and the following relationships hold:

$$x(t) = \theta_1(t) - \theta_{10} = x(0)\exp(t\lambda); \quad (18)$$
$$\lambda = \frac{|\Gamma_2|}{\pi R^2}\frac{\sqrt{D}}{\sin^2(\theta_{20} - \theta_{10})\sin\theta_{20}\sin\theta_{10}}$$

According to (17), instability with respect to extremely small disturbances takes place when inequalities (17) are violated for $|\Gamma_1/\Gamma_2| > 5.24$ or $|\Gamma_1/\Gamma_2| < 1.35$. For example, if $|\Gamma_1/\Gamma_2| \approx 1.3$ in (18) then $\sqrt{D} \approx 0.14$ that allows estimating the estimating the character time of the disturbance growth $\lambda^{-1} \approx 14 days$ if accepting that anti-cyclonic APV intensity is $|\Gamma_2| \approx RU; U \approx 10 m/\sec$, $R \approx 6371 km$.

It means that in the period of about one month even in the case of small disturbances with amplitude about 1^0 the deviation from the equilibrium up to 5^0 is possible with manifestation of a dynamic mode like in Fig. 1a.

Results of comparison of the stability condition (16) with observations [15] are presented below. An analysis was conducted in [15] for the variability in the mean distance between ACAs by longitude, Δλ for the Atlantic (Icelandic Low and Azores High) and Pacific (Aleutian Low and Hawaiian High) ACAs as abnormality characteristics of the position and instability of the vortex pairs. In particular, reanalysis data were used for the detection of ACA characteristics similar to [17]. The abnormality (instability) of mutual ACA positioning during the particular season was characterized in [15] by deviation from the long-term mean against the standard deviation. Also the degree of abnormality (instability) in the temperature difference between land and ocean ΔT in the Northern Hemisphere was estimated with respect to the mean conditions from CRU data for winters (http://www.uea.ac.uk/cru/data). The obtained estimates show the comparability of the dynamic and thermal factors in the formation of stability modes or instability of the ACA mutual positioning on the sphere.

Fig. 2 and Fig. 3 show stability regions depending on the positions of anticyclonic and cyclonic vortices according to condition (16). The horizontal axis on Figs. 2 and Fig. 3 corresponds to the co-latitude θ for the anticyclonic ACA vortex center, and the vertical axis for that of the cyclonic ACA. Crosses on figures characterize mean values of θ for the respective vortices of the ACA pair (by the cross position) and their standard deviations (by the cross size in the respective direction). Dark region shows stability obtained according to the condition (16). The stationary mode of the vortex pair on Fig. 2 (similar to Icelandic-Azores ACA pair in 1988) falls in the stability region, and on Fig. 3 (similar to Icelandic-Azores ACA pair in 1964) does not fall into it. Shadowed regions on Figs. 2 and 3 correspond to the conditions of realization of exactly anti-cyclonic circulation of the polar vortex in the Northern Hemisphere when the right hand side of (14) and (15) is positive.

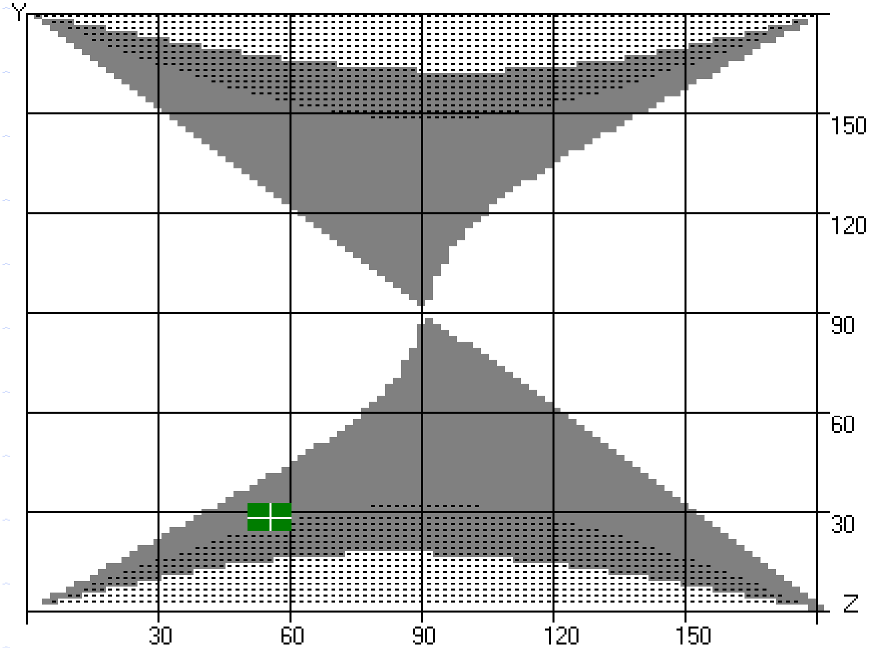

Fig. 2 Stability regions for $\gamma_1 = \frac{\Gamma_1}{\Gamma_2} = -1.87$ with co-latitudinal position of vortex pair at $\theta_{10} = 30^0; \theta_{20} = 57.5^0$ in the stability area.

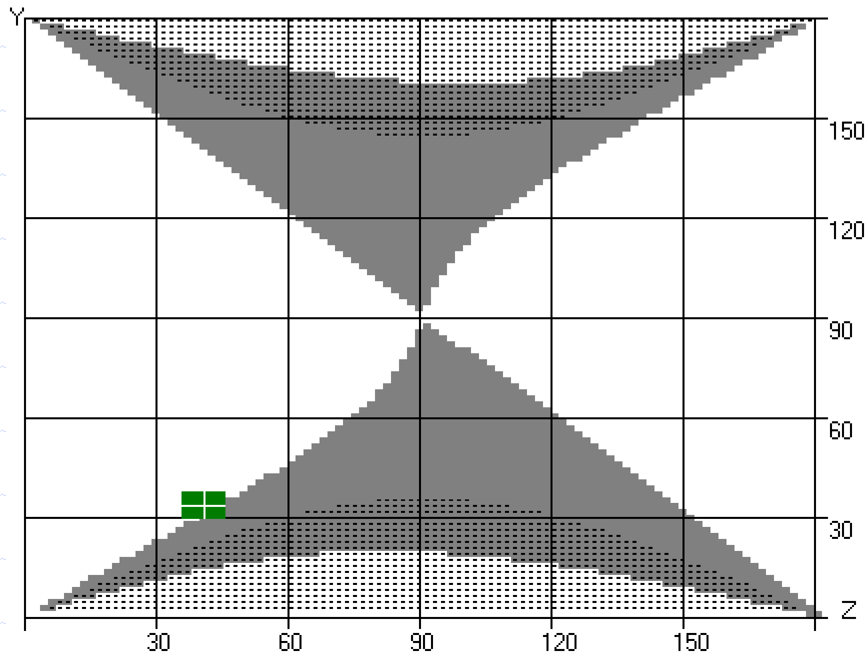

Fig. 3. Stability regions for $\gamma_1 = \frac{\Gamma_1}{\Gamma_2} = -1.71$ with co-latitudinal position of vortex pair at $\theta_{10} = 35^0; \theta_{20} = 42.5^0$ in the instability area.

On the whole, the comparison of the theoretical results with long-term observational data [15] indicates their general agreement.

6. Conclusion

It is shown that only APV meets the hydrodynamic equations on a sphere. The stream functions used in [3], [8]-[11] do not meet the hydrodynamic equations (1), so it cannot be realized in fluid dynamics. Thus, APV is the only correct elementary vortex object on a sphere.

An exact solution for the APVs dynamics on the rotating sphere is obtained by using stream function of solid-state rotation, corresponding to zero absolute vorticity. On this base a steady solution for two APVs is obtained which gives a possible hydrodynamic mechanism for long-living stable cyclonic-anticyclonic vortex pairs like IC and AA and their dynamics. The possible sensitivity of IC and AA centers of atmospheric action over North Atlantic to the different type of initial disturbances is also explained on the base of linear and nonlinear stability analysis of the steady solution obtained.

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REFERENCES

1. I. S. Gromeka, “Collection of works,” Moscow, USSR AS Publ. (1952) (in Russian)
2. V. A. Bogomolov, “Dynamics of vorticity on a sphere,” Fluid Dyn. **6**, 863 (1977)
3. V. A. Bogomolov, “Model of oscillations of atmosphere action centers” , Isv. Acad. Sci. USSR Atmos. Oceanic Phys. **15**, 18 (1979)
4. A. V. Borisov, A. A. Kilin, and I. S. Mamaev, “New integrable problem on vortexes point motion on a sphere,” Nonlin. Dyn. **3**, 211 (2007)
5. V. A. Bogomolov, On the motion of a vortex on a rotating sphere, Izvestiya Atmos. Ocean Phys., **21**, 298 (1985)
6. K. V. Klyatskin and G. M. Reznik, Point vortices on a rotating sphere, Oceanology, **29**, 12 (1989)
7. P. K. Newton and H. Shokraneh, The N-vortex problem on a rotating sphere: I Multi-frequency configurations, Proc. Roy. Soc. A**462**, 149 (2006)
8. Y. Kimura and H. Okamoto, “Vortex motion on a sphere,” J. Phys. Soc. Japan, **56**, 4203 (1987)
9. D. G. Dritschel and S. Boatto, “The motion of point vortices on closed surfaces,” Proc. R. Soc. A. **471**, 20140890 (2015)
10. J. Andrade, S. Boatto, and C. Vidal, Dynamics and bifurcation of passive tracers advected by a ring of point vortices on a sphere, J. Math. Phys., **61**(5), 052702 (2020)

11. S. Boatto and C. Sino, A vortex ring on a sphere: the case of total vorticity equal to zero, Phil. Trans. Roy. Soc. A, **377**(2158), 20190019 (2019)
12. S. G. Chefranov, I. I. Mokhov, and A. G. Chefranov, The hydrodynamic singular vortex on the sphere and the Dirac monopole, arXiv: 1711.04124v1 [physics.flu-dyn] 11 Nov 2017
13. G. K. Batchelor, "An Introduction to Fluid Dynamics," Cambridge Univ. Press, Cambridge (1967)
14. I. I. Mokhov, S. G. Chefranov, and A. G. Chefranov, "Dynamics of delta-singular vortices on a rotating sphere and stability of paired atmospheric centers of action," Doklady Earth Sci. **433** (1), 948 (2010)
15. I. I. Mokhov, S. G. Chefranov, and A. G. Chefranov, "Interaction of global–scale atmospheric vortices: Modeling based on Hamiltonian dynamic system of antipodal point vortices on rotating sphere," Procedia IUTAM **8**, 176 (2013); arXiv: 1212.1550v1[phys.flu-dyn] 7 Dec 2012
16. I. I. Mokhov, S. G. Chefranov, and A. G. Chefranov, "Interaction between global-scale atmospheric vortices: Modeling with Hamiltonian dynamic system of antipodal point vortices on a rotating sphere", arXiv: 1611.01963v2 [physics.flu-dyn] 5 Oct 2017
17. I. I. Mokhov and V. C. Khon, "Interannual variability and long-term tendencies of change in atmospheric centers of action in the Northern Hemisphere: Analysis of observational data," Izvestiya, Atmos. Oceanic Phys. **41** (6), 723 (2005)