Deflection of a light ray passing through an oscillating dark matter halo

Vladimir Koutvitsky and Eugene Maslov

IZMIRAN, Russia

XXIX Scientific session of the RAS Council on Nonlinear Dynamics. Moscow, Dec. 14-15, 2020

According to the basic concepts of General Relativity, fotons in a curved spacetime move along null geodesics $x^\mu = x^\mu(\lambda)$ satisfying the equation

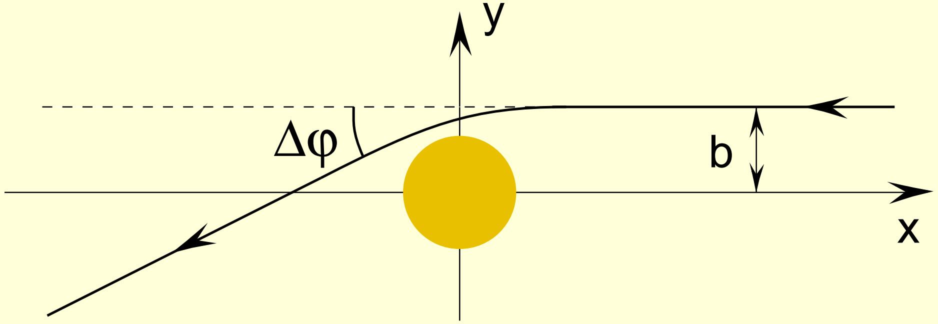
$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0,$$

where λ is an affine parameter. This equation determines all geodesic characteristics, including the deflection angle when passing near a gravitating mass.

For **static** asymptotically flat spacetimes the deflection angle is given by

$$\Delta\varphi = \frac{4GM}{b} + O((r_g/b)^2),$$

where G is the gravitational constant, M is the total gravitating mass, b is the impact parameter, $r_g = 2GM$. For the ray passing in the vicinity of the solar limb $\Delta\varphi \approx 1.75''$.



Deflection of light in time-dependent metrics

By the localized source of gravitational waves:

- T. Damour and G. Esposito-Farése, PRD (1998);
- S. M. Kopeikin, G. Schäfer, C. R. Gwinn, and T. M. Eubanks, PRD (1999).

The effect of gravitational waves appears only in high orders of expansion in 1/b.

An analytical study of light propagation in the gravitational field of an ensemble of arbitrarily moving and spinning point-like masses using retarded Liénard-Wiechert potentials:

- S. M. Kopeikin and G. Schäfer, PRD(1999);
- S. Kopeikin and B. Mashhoon, PRD (2002).

Effect of cosmological expansion on light ray deflection (using McVittie metric):

- O. F. Piattella, Universe (2016)

No effect of cosmological background on the deflection angle in the leading order.

Deflection of light by localized scalar field configurations

- K. S. Virbhadra, D. Narasimha, and S. M. Chitre, A&A (1998).

The effect of the static spherically symmetric distribution of a massless scalar field studied on the basis of the Janis-Newman-Winicour solution.

- M. P. Dąbrowski and F. E. Schunck, ApJ (2000).
- F. E. Schunck, B. Fuchs, and E. W. Mielke, MNRAS (2006).

The gravitational lensing by a static spherically symmetric halo constructed from the nonlinear complex scalar field without and with a ϕ^6-type self-interaction.

- M. Bošković, F. Duque, M. C. Ferreira, F. S. Miguel,and V. Cardoso, PRD (2018).

A numerical study of motion of test particles in time-dependent gravitational fields of oscillating configurations of a non-self-interacting scalar field:

Deflection of light in nonstatic spherically symmetric gravitational fields

Consider a spherically symmetric **nonstatic** metric of the form

$$ds^2 = B(t,r)\,dt^2 - A(t,r)\,dr^2 - r^2(d\vartheta^2 + \sin^2\vartheta\,d\varphi^2).$$

For light rays lying in the plane $\vartheta = \pi/2$, the geodesic equation reduces to

$$\frac{d}{d\lambda}\ln\left(B\frac{dt}{d\lambda}\right) = \frac{\dot{B}}{2B}\frac{dt}{d\lambda} - \frac{\dot{A}}{2B}\left(\frac{dr}{d\lambda}\right)^2\left(\frac{dt}{d\lambda}\right)^{-1},$$

$$\frac{d^2r}{d\lambda^2} + \frac{B'}{2A}\left(\frac{dt}{d\lambda}\right)^2 + \frac{\dot{A}}{A}\frac{dt}{d\lambda}\frac{dr}{d\lambda} + \frac{A'}{2A}\left(\frac{dr}{d\lambda}\right)^2 - \frac{r}{A}\left(\frac{d\varphi}{d\lambda}\right)^2 = 0,$$

$$\frac{d^2\varphi}{d\lambda^2} + \frac{2}{r}\frac{dr}{d\lambda}\frac{d\varphi}{d\lambda} = 0,$$

where $(\cdot) = \partial/\partial t$, $(') = \partial/\partial r$. For a ray coming from infinity with impact parameter b

$$\frac{d\varphi}{d\lambda} = \frac{b}{r^2}, \qquad B\left(\frac{dt}{d\lambda}\right)^2 - A\left(\frac{dr}{d\lambda}\right)^2 - \frac{b^2}{r^2} = 0.$$

In the weak field approximation

$$A = 1 - 2\psi + O(\varkappa^2), \quad B = 1 + 2\chi + O(\varkappa^2),$$

where $\psi(t,r)$ and $\chi(t,r)$ are time-periodic functions of order $\varkappa \ll 1$, and $\varkappa \sim G$.

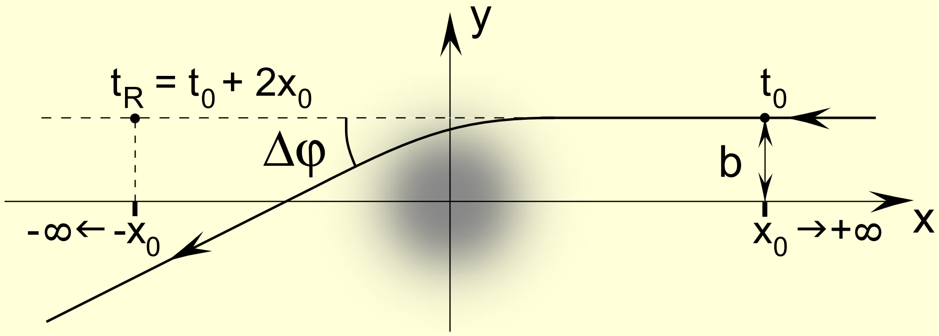
Trajectory without gravitating mass (straight line):

$$x = x_0 + t_0 - t, \quad y = b,$$

$$r(t) = \sqrt{x^2(t) + b^2}, \qquad t = \lambda.$$

With gravitating mass (deflected trajectory):

$$r(t) = (1 + \eta(t))\sqrt{x^2(t) + b^2}, \qquad B\frac{dt}{d\lambda} = 1 + \zeta(t) \qquad (\eta \sim \zeta \sim \varkappa \ll 1).$$



From the geodesic equations we obtain

$$\frac{d\zeta}{dt} = \dot{\chi}(t,r) + \dot{\psi}(t,r)\left(1 - \frac{b^2}{r^2}\right),$$

$$x(x^2+b^2)\frac{d\eta}{dt} - (x^2-b^2)\eta + x^2\psi(t,r) + (x^2-b^2)\chi(t,r) + b^2\zeta(t) = 0.$$

$$\frac{d\varphi}{dt} = \frac{b}{x^2+b^2}\left[1 + (2\chi - \zeta - 2\eta)\right].$$

Integration of these equations gives

$$\zeta = \int_x^\infty \left[\dot{\chi}(t,r) + \dot{\psi}(t,r)\left(1 - \frac{b^2}{r^2}\right)\right] dx,$$

$$\eta = \frac{x}{x^2+b^2}\left\{\int \left[x^2\psi(t,r) + (x^2-b^2)\chi(t,r) + b^2\zeta(t)\right]\frac{dx}{x^2} + const\right\},$$

$$\varphi = \pi + \Delta\varphi,$$

where the deflection angle is

$$\Delta\varphi = b\int_{-\infty}^{\infty} \frac{2\chi - \zeta - 2\eta}{x^2+b^2}\, dx.$$

The obtained formula for the deflection angle,

$$\Delta\varphi = b\int_{-\infty}^{\infty} \frac{2\chi - \zeta - 2\eta}{x^2 + b^2}\, dx,$$

is valid not only for time-periodic metrics, but also for static ones. In this case $\zeta = 0$.

Consider, for example, the Schwarzschild metric. Assuming $r_g/b = \varkappa \ll 1$, where $r_g = 2GM$ is the gravitational radius, we have

$$\psi = \chi = -\varkappa\frac{b}{2r}.$$

$$\eta = -\varkappa\frac{bx}{x^2 + b^2}\left(\frac{\sqrt{x^2 + b^2}}{2x} + \operatorname{arsh}\frac{x}{b} + const\right).$$

Integration gives the well-known result

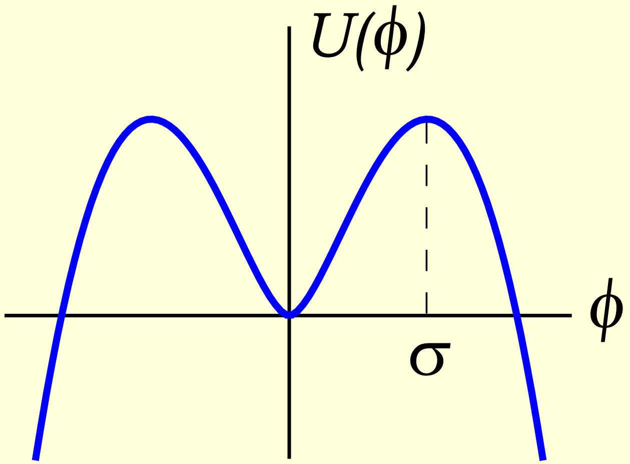
$$\Delta\varphi = \frac{4GM}{b} + O((r_g/b)^2).$$

In the case of a time-periodic metric, the deflection angle will generally depend on the photon emission time t_0 or, which is the same, on the observation time $t_R = t_0 + 2x_0$.

Deflection of light by a time-periodic spherically symmetric scalar field

As a deflecting mass, we consider a pulsating dark matter halo made from the self-gravitating real scalar field with the potential

$$U(\phi) = \frac{1}{2}m^2\phi^2\left(1 - \ln\frac{\phi^2}{\sigma^2}\right)$$



● quantum field theory [G. Rosen (1969), Bialynicki-Birula & Mycielski (1975)]

● inflationary cosmology [Linde (1982, 1992), Albrecht & Steinhardt (1982), Barrow & Parsons (1995)]

● supersymmetric extensions of the Standard Model (flat direction potentials in the gravity mediated supersymmetric breaking scenario) [Enqvist & McDonald (1998)]

Einstein-Klein-Gordon system

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G\left[\phi_{,\mu}\phi_{,\nu} - \left(\frac{1}{2}\phi_{,\alpha}\phi^{,\alpha} - U(\phi)\right)g_{\mu\nu}\right],$$

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x_\mu}\left(\sqrt{-g}\frac{\partial\phi}{\partial x^\mu}\right) + \frac{dU(\phi)}{d\phi} = 0.$$

The case of spherical symmetry: $ds^2 = Bdt^2 - Adr^2 - r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)$:

$$\frac{A_r}{A} + \frac{A-1}{r} = 4\pi GrA\left[\frac{1}{B}\phi_t^2 + \frac{1}{A}\phi_r^2 + m^2\phi^2\left(1 - \ln\frac{\phi^2}{\sigma^2}\right)\right],$$

$$\frac{B_r}{B} - \frac{A-1}{r} = 4\pi GrA\left[\frac{1}{B}\phi_t^2 + \frac{1}{A}\phi_r^2 - m^2\phi^2\left(1 - \ln\frac{\phi^2}{\sigma^2}\right)\right],$$

$$\frac{1}{B}\phi_{tt} - \frac{1}{A}\left(\phi_{rr} + \frac{2}{r}\phi_r\right) + \frac{1}{2B}\left(\frac{A_t}{A} - \frac{B_t}{B}\right)\phi_t + \frac{1}{2A}\left(\frac{A_r}{A} - \frac{B_r}{B}\right)\phi_r = m^2\phi\ln\frac{\phi^2}{\sigma^2}.$$

Boundary conditions

$$\phi(t,\infty) = 0,\ A(t,\infty) = 1,\ B(t,\infty) = 1,\ \phi_r(t,0) = 0,\ A(t,0) = 1.$$

This system has a pulsating solution of the form

$$\phi(t,r) = \sigma[a(\theta) + \varkappa Q(\theta,\rho) + O(\varkappa^2)]e^{(3-\rho^2)/2},$$

$$A(t,r) = \left(1 - \frac{\rho_g}{\rho}\right)^{-1}, \quad B(t,r) = \left(1 - \frac{\rho_g}{\rho}\right) e^{-s},$$

where

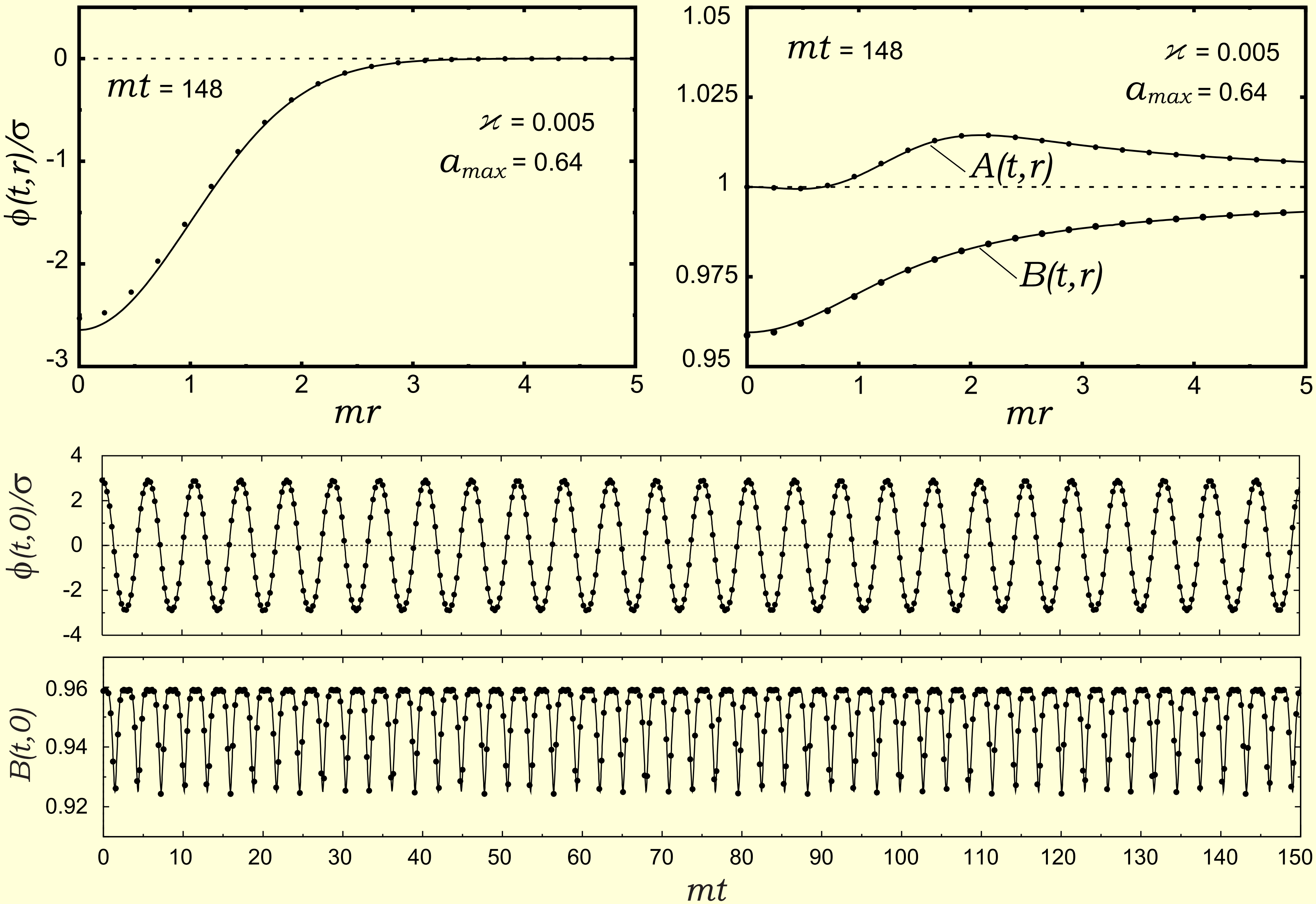
$$\rho_g(\tau,\rho) = -\varkappa\rho\left[V_{\max}\left(1 - \frac{\sqrt{\pi}\,\mathrm{erf}\rho}{2\rho}e^{\rho^2}\right) + a^2\rho^2\right]e^{3-\rho^2} + O(\varkappa^2),$$

$$s(\tau,\rho) = \varkappa(2V_{\max} + a^2\ln a^2 + a^2\rho^2)e^{3-\rho^2} + O(\varkappa^2),$$

$\tau = mt$, $\rho = mr$, $\varkappa = 4\pi G\sigma^2 \ll 1$ (G is the gravitational constant). The function $a(\theta(\tau))$ oscillates in the range $-a_{\max} \leqslant a(\theta) \leqslant a_{\max}$ in the local minimum of the potential $V(a)$:

$$a_{\theta\theta} = -dV/da, \qquad V(a) = (a^2/2)\left(1 - \ln a^2\right) \leqslant V_{\max} = V(a_{\max}),$$

where $\theta_\tau = 1 + \varkappa\Omega + O(\varkappa^2)$, and the constant $\varkappa\Omega$ is the pulson frequency correction due to gravitational effects. The function $Q(\theta,\rho)$ is a series in Hermite polynomials whose coefficients are periodic (in θ) solutions of nonhomogeneous Hill equations.



0
-1
-2
-3
ϕ(t,r)/σ
mt = 148
ϰ = 0.005
a_max = 0.64
0 1 2 3 4 5
mr
1.05
1.025
1
0.975
0.95
mt = 148
ϰ = 0.005
a_max = 0.64
A(t,r)
B(t,r)
0 1 2 3 4 5
mr
4
2
0
-2
-4
ϕ(t,0)/σ
0.96
0.94
0.92
B(t,0)
0 10 20 30 40 50 60 70 80 90 100 110 120 130 140 150
mt

Since the metric found is everywhere regular and has no horizon, we can rewrite the functions $A(t,r)$ and $B(t,r)$ with the required accuracy in the form

$$A = 1 - 2\psi + O(\varkappa^2), \quad B = 1 + 2\chi + O(\varkappa^2),$$

where

$$\psi(t,r) = \frac{\varkappa}{2}\left[V_{\max}\left(1 - \frac{\sqrt{\pi}\,\mathrm{erf}\rho}{2\rho}e^{\rho^2}\right) + a^2\rho^2\right]e^{3-\rho^2},$$

$$\chi(t,r) = -\frac{\varkappa}{2}\left[V_{\max}\left(1 + \frac{\sqrt{\pi}\,\mathrm{erf}\rho}{2\rho}e^{\rho^2}\right) + a^2\ln a^2\right]e^{3-\rho^2}.$$

Calculating $\dot{\psi}(t,r)$, $\dot{\chi}(t,r)$ and setting

$$\begin{aligned} \tau &= \tau_R - \xi_0 - \xi, \quad \xi = mx, \quad \tau_R = mt_R, \quad \xi_0 = mx_0 \to \infty \\ \beta &= mb, \qquad \rho^2 = \xi^2 + \beta^2, \qquad d/d\tau = -d/d\xi, \end{aligned}$$

we find

$$\zeta = \frac{\varkappa}{2}e^{3-\beta^2}\int_{\xi}^{\infty}\left[\frac{d}{d\xi}\left(a^2\ln a^2\right) - \xi^2\frac{d}{d\xi}a^2\right]e^{-\xi^2}d\xi.$$

On the other hand,

$$\frac{d^2a^2}{d\xi^2} = \frac{d^2a^2}{d\tau^2} = \frac{d^2a^2}{d\theta^2}\theta_\tau^2 = 4V_{\max} - 2a^2 + 4a^2\ln a^2 + O(\varkappa).$$

Using these relations and integrating by parts, we finally obtain

$$\zeta = -\frac{\varkappa}{4}e^{3-\rho^2}\left(\frac{1}{2}\frac{d^2}{d\xi^2} + \xi\frac{d}{d\xi}\right)a^2 + O(\varkappa^2).$$

Now we substitute ψ, χ and ζ into the formula for η and integrate over ξ. This gives

$$\begin{aligned}\eta &= \frac{\varkappa}{2}e^{3-\beta^2}\left\{\sqrt{\pi}\,V_{\max}\left[\frac{\xi\,\mathrm{erf}\,\xi}{\rho^2} - e^{\beta^2}\left(\frac{\xi}{\rho^2}\int_0^\xi \frac{\mathrm{erf}\rho}{\rho}\,d\xi + \frac{\mathrm{erf}\rho}{2\rho}\right)\right]\right.\\ &\left. -\frac{1}{2}e^{-\xi^2}\left(a^2 + \frac{\xi}{2\rho^2}\frac{da^2}{d\xi}\right) + const\,\frac{\xi}{\rho^2}\right\} + O(\varkappa^2).\end{aligned}$$

Now we rewrite the general expression for the deflection angle as

$$\Delta\varphi = \beta \int_{-\infty}^{\infty} \frac{2\chi - \zeta - 2\eta}{\xi^2 + \beta^2}\, d\xi$$

and substitute there

$$2\chi - \zeta - 2\eta = \varkappa e^{3-\beta^2} \left\{ \sqrt{\pi}\, V_{\max} \frac{\xi}{\rho^2} \left[e^{\beta^2} \int_0^{\xi} \frac{\operatorname{erf} \rho}{\rho}\, d\xi - \operatorname{erf} \xi \right] \right.$$

$$\left. - \frac{1}{8} \frac{d^2 a^2}{d\xi^2} e^{-\xi^2} + \frac{1}{4} \left(1 + \frac{1}{\rho^2} \right) \frac{d a^2}{d\xi} \xi e^{-\xi^2} + const\, \frac{\xi}{\rho^2} \right\}.$$

It is remarkable that the last three terms in this formula, two of which contain derivatives of the oscillating function a^2, do not contribute to the integral in $\Delta\varphi$.

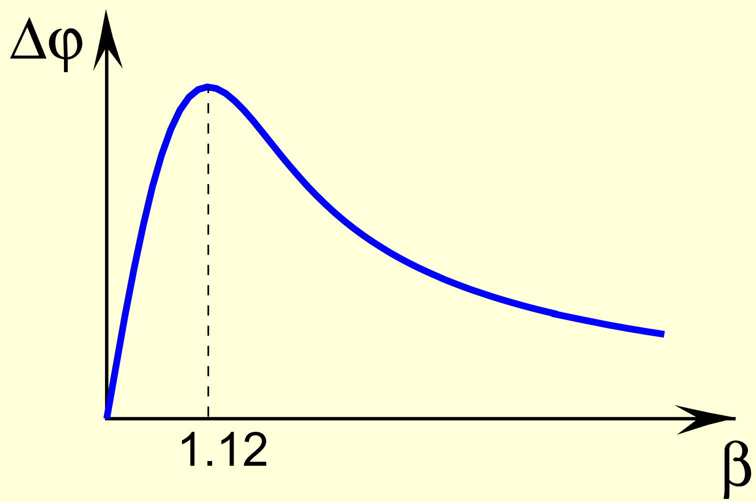
As a result, after integrating, we arrive at a simple formula

$$\begin{aligned}\Delta\varphi &= \varkappa\frac{e^3\sqrt{\pi}V_{\max}}{\beta}\left(1-e^{-\beta^2}\right)+O\left(\varkappa^2\right)\\ &= \frac{4GM}{b}\left(1-e^{-m^2b^2}\right)+O\left(\varkappa^2\right),\end{aligned}$$

where b is the impact parameter, M is the halo mass,

$$M=\left(e\sqrt{\pi}\right)^3\sigma^2 m^{-1}V_{\max}\left(1+O(\varkappa)\right).$$

It is interesting that the deflection angle is time-independent in the leading order, despite the scalar field oscillations. This is a specific feature of the logarithmic potential. The maximum value of $\Delta\varphi$ is achieved at $mb=1.1209$.




Thank you
for your attention!