Длинные нелинейные стоячие волны в протяженном бассейне с пологими берегами: теория и эксперимент

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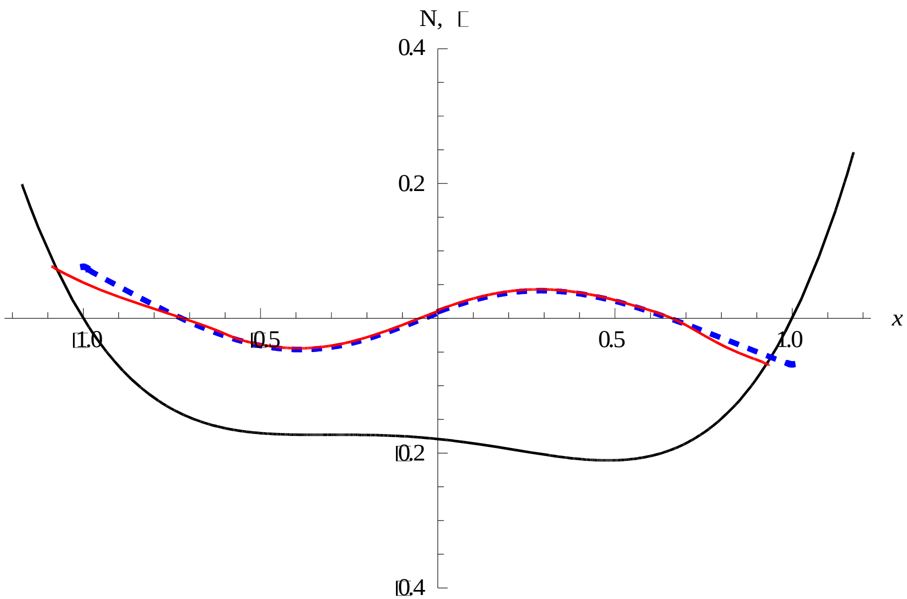
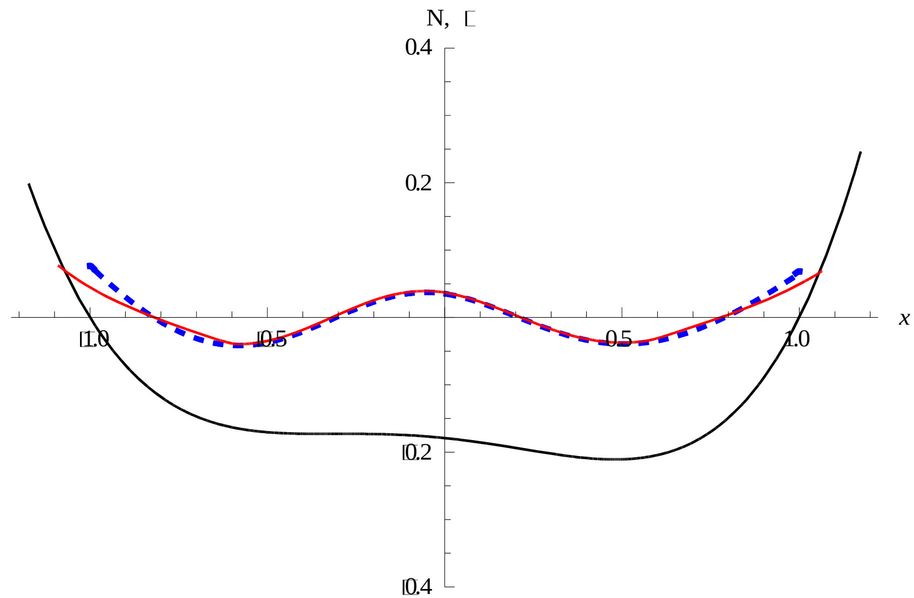
Shallow water: two shores

$$\eta_t + \big((D(x)+\eta)u\big)_x = 0, \quad u_t + g\eta_x + uu_x = 0.$$

Depth: $D(a) = D(b) = 0 \quad D'(a) \neq 0, \quad D'(b) \neq 0$

Boundary condition at two variable boundaries x(t): $\eta(x_a(t), t) + x_a(t) = 0$





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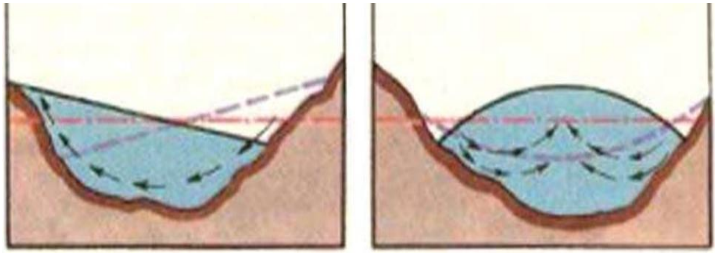
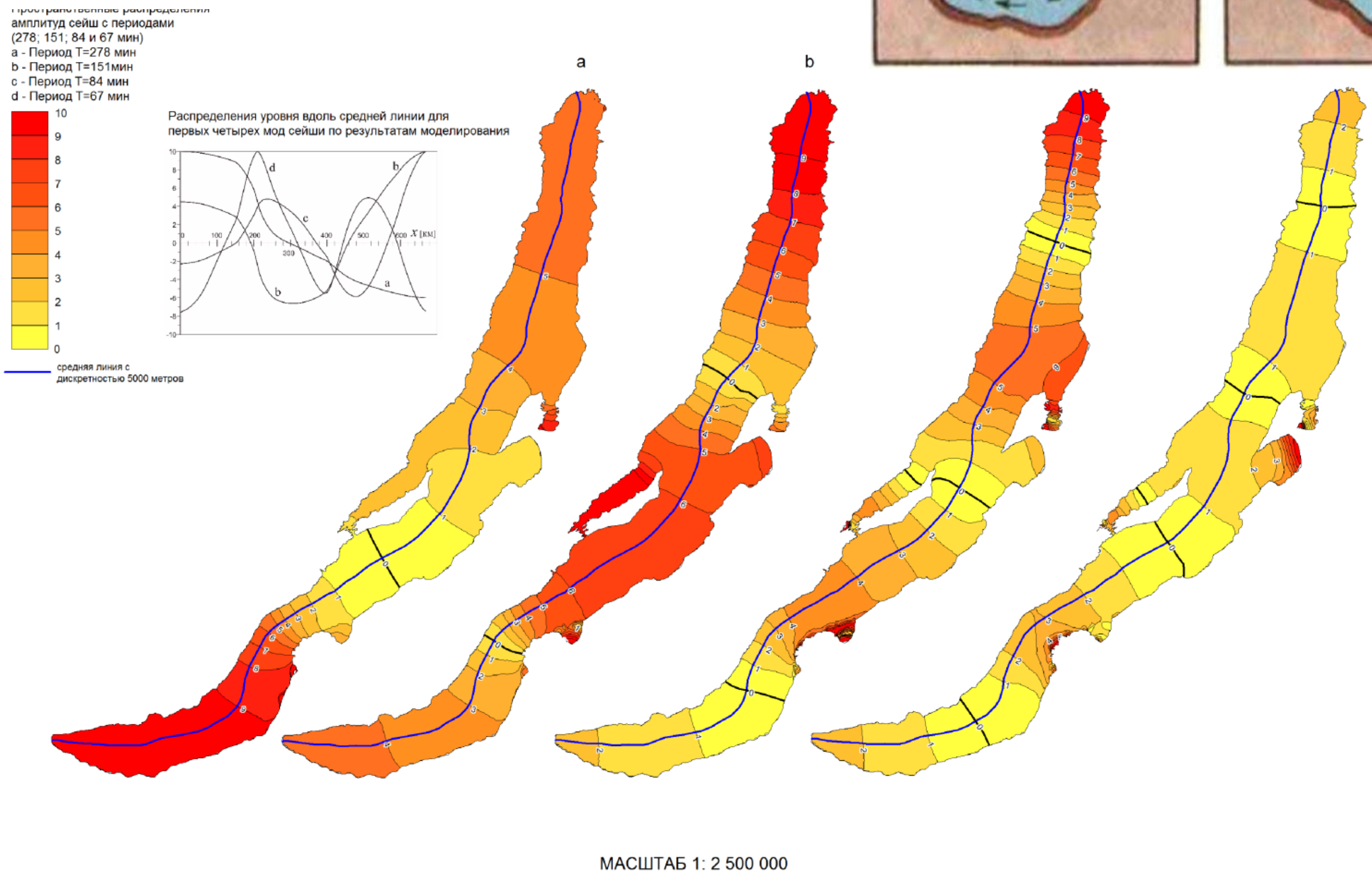
Motivation: Seiches – standing waves

A ***seiche*** is a standing wave in an enclosed or partially enclosed body of water.

One- and two-nodes seiches



The Baikal Lake seiches



Linearized shallow water: two shores

$$N_\tau + (D(y)U)_y = 0, \quad U_\tau + N_y = 0$$

Depth: $D(a) = D(b) = 0$

$D'(a) \neq 0, \quad D'(b) \neq 0$

No boundary conditions,

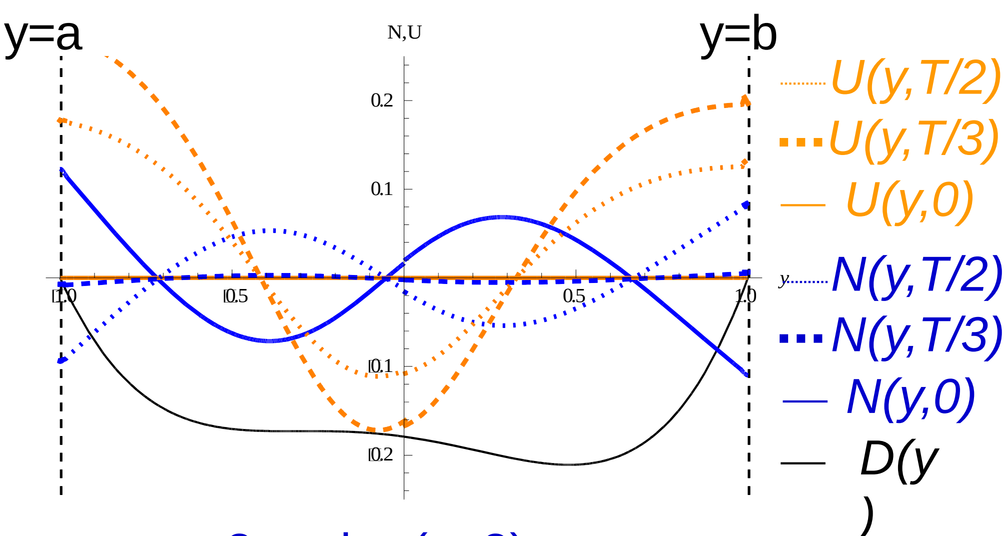
Finite energy condition: $E^2 = \|N\|^2 + \|\sqrt{D(y)}\,U\|^2 < \infty$

Asymptotics. Phase: $S(x,y) = \int_x^y \frac{dy}{\sqrt{D(y)}}, \quad a \le x \le y \le b.$

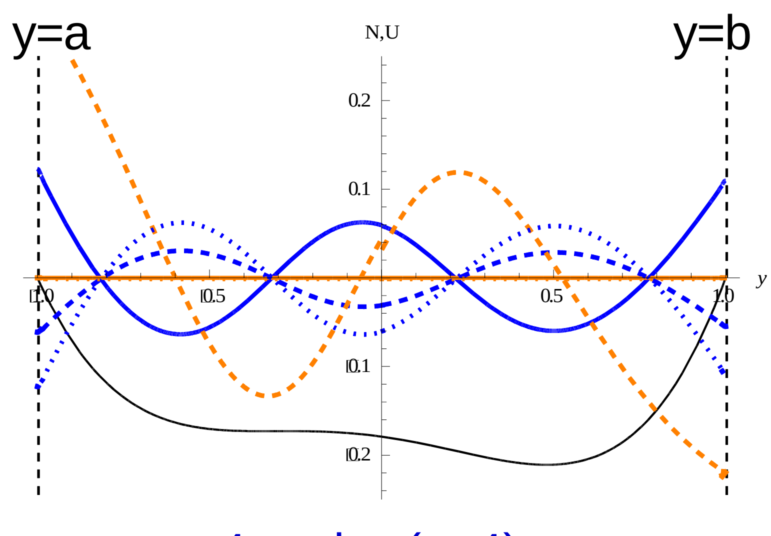
Quantization: $w_n = \frac{\pi}{S(a,b)}(\frac{1}{2} + n)(1 + O(n^{-1})),\ n \in \mathbb{N}.$

$$N_a(y,\tau) = c\cos(w_n\tau + \tau_0)\mathbf{J}_0(w_n S(a,y))\left(\frac{S(a,y)}{\sqrt{D(y)}}\right)^{1/2} \qquad y \in [a, b-\delta]$$

$$N_b(y,\tau) = c(-1)^n\cos(w_n\tau + \tau_0)\mathbf{J}_0(w_n S(y,b))\left(\frac{S(y,b)}{\sqrt{D(y)}}\right)^{1/2} \qquad y \in [a+\delta, b]$$



3-nodes (n=3)



4-nodes (n=4)

Linearized shallow water: shore and vertical wall

$$N_\tau + (D(y)U)_y = 0, \quad U_\tau + N_y = 0$$

Depth: $D(a) = 0, D'(a) \neq 0$

Boundary condition at y=b: $U(b,\tau) = 0$

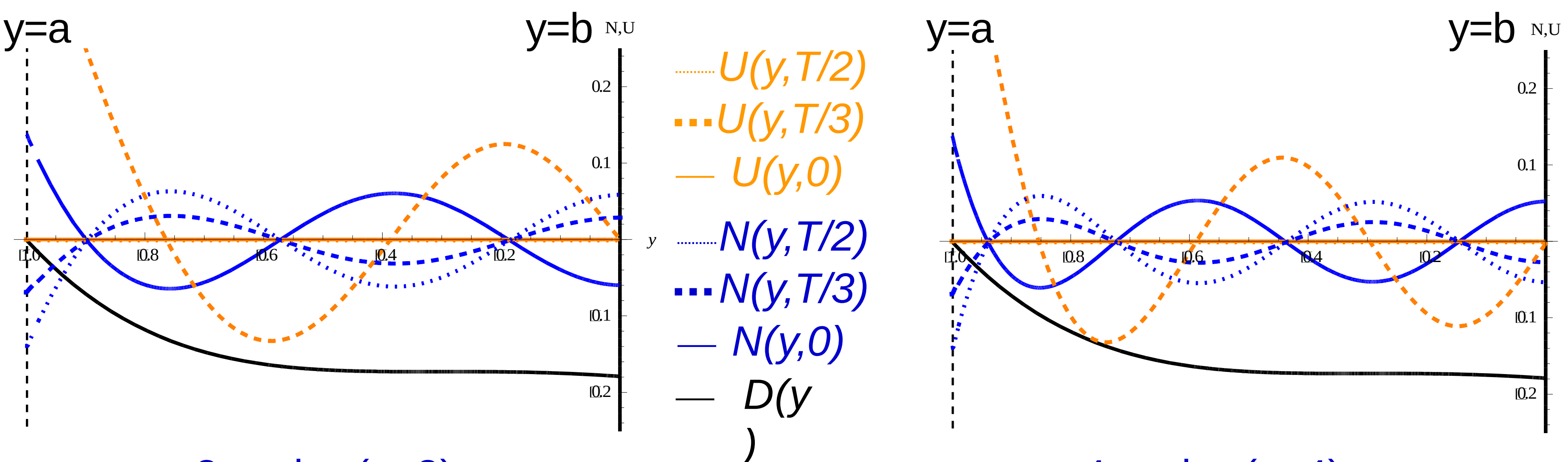
Finite energy condition: $E^2 = \|N\|^2 + \|\sqrt{D(y)}\, U\|^2 < \infty$

Asymptotics. Phase: $S(x,y) = \int_x^y \frac{dy}{\sqrt{D(y)}}, \quad a \le x \le y \le b.$

Quantization: $w_n = \frac{\mu_n}{S(a,b)} \approx \frac{\pi}{S(a,b)}\left(\frac{1}{4} + n\right) \qquad \mathbf{J}_1(\mu_n) = 0, \quad n \in \mathbb{N}$

$$N_a(y,\tau) = c\cos(w_n\tau + \tau_0)\mathbf{J}_0(w_n S(a,y)) \left(\frac{S(a,y)}{\sqrt{D(y)}}\right)^{1/2} (1 + O(w_n^{-3/2})),$$

$$U_a(y,\tau) = c\sin(w_n\tau + \tau_0)\mathbf{J}_1(w_n S(a,y)) \left(\frac{S(a,y)}{\sqrt{D(y)}}\right)^{1/2} \frac{1}{\sqrt{D(y)}} (1 + O(w_n^{-3/2}))$$



3-nodes (n=3)

4-nodes (n=4)

The main defect in the linear model:
it does not describe the splash (run up)

$\Longrightarrow$ Carrier-Greenspan transform or its asymptotic modification near the beach

Shallow water: slopping bottom, Carrier—Greenspan

$$\eta_t + ((D(x)+\eta)u)_x = 0, \quad u_t + g\eta_x + uu_x = 0$$

Depth $D(x) = \gamma(x-a)$:

Boundary condition at left variable boundary x(t): $\eta(x_a(t),t) + x_a(t) = 0$

Boundary condition at x=b: $u|_{x=b} = 0$

Withount loss of generality: g=1, a=0, b=1, $\gamma = 1$

Carrier—Greenspan transform.

$$t = \tau + U, \quad x = y - N + U^2/2, \quad \eta = N - U^2/2, \quad u = U \iff$$

$$\tau = t - u, \quad y = x + \eta - u^2/2, \quad N = \eta + u^2/2, \quad U = u.$$

$$J \equiv \frac{\partial(\tau,y)}{\partial(t,x)} = 1 + \eta_x - u_t - \eta_x u_t + \eta_t u_x, \quad J^{-1} \equiv \frac{\partial(t,x)}{\partial(\tau,y)} = 1 - N_y + U_\tau - N_y U_\tau + N_\tau U_y + UU_y$$

Theorem (C—G).

If $J > 0$, $J < \infty$
then shallow water
is equivalent to linearized SW:

$$\begin{pmatrix} \dfrac{\partial v}{\partial t} + \dfrac{\partial[\eta + v^2/2]}{\partial x} \\ \dfrac{\partial \eta}{\partial t} + \dfrac{\partial[(\eta+x)v]}{\partial x} \end{pmatrix} = \frac{\partial(\tau,y)}{\partial(t,x)} \begin{pmatrix} \dfrac{\partial U}{\partial \tau} + \dfrac{\partial N}{\partial y} \\ \dfrac{\partial N}{\partial \tau} + \dfrac{\partial[yU]}{\partial y} \end{pmatrix} = 0$$

$x_a(t)$ becomes fixed boundary: $y_a(t)=0$. Finite energy condition or smth alike is required.

Boundary condition at x=b becomes nonlin: $U(Y(\tau),\tau) = 0, \quad Y(\tau) = b + N(Y(\tau),\tau)$

Reduced Carrier-Greenspan transformation (simplification):

$$D(y) = D(x) + \eta(x,t)\rho(x),\ \tau = t,\ N(y,\tau) = \eta(x,t),\ U(y,\tau) = u(x,t),$$

here $\rho(x)$ is a cut-off function, $\rho = 1$ near x^0 : $\quad D(x) = 0$ and $\rho = 0$ outside of the neighborhood of the point x^0.

The main property: **the boundary becomes fixed**

Shallow water: two shores

$$\eta_t + \left((D(x)+\eta)u\right)_x = 0, \qquad u_t + g\eta_x + uu_x = 0$$

Depth: $D(a) = D(b) = 0 \qquad D'(a) \neq 0, \quad D'(b) \neq 0$

Boundary condition at two variable boundaries x(t): $\eta(x_a(t), t) + x_a(t) = 0$

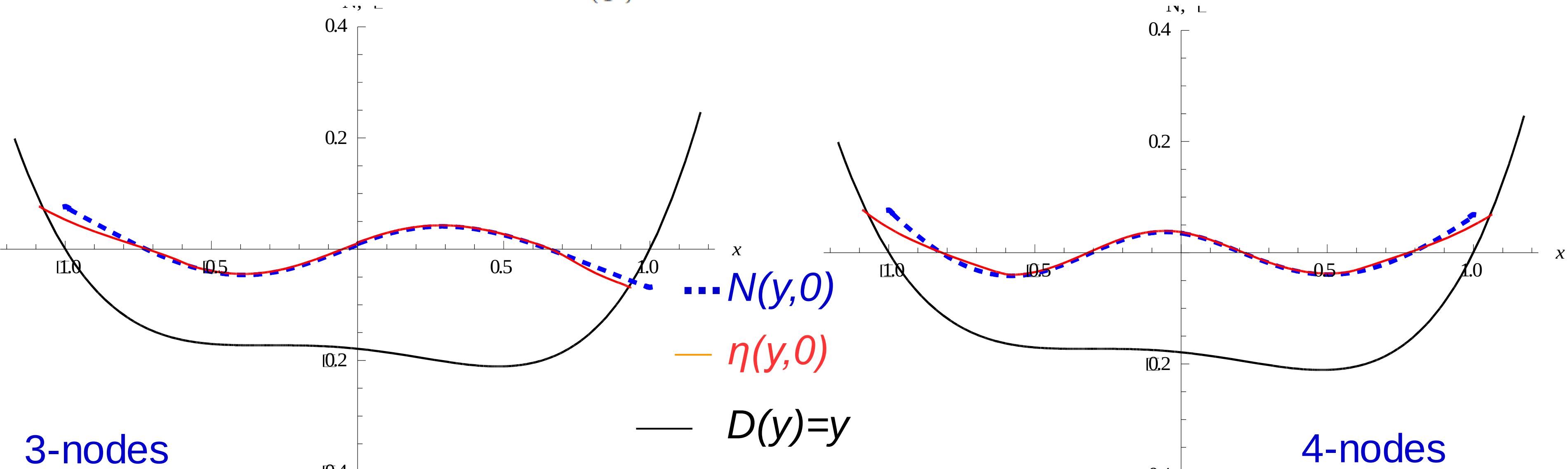
Reduced Carrier—Greenspan transform with cutting function ρ:

$$D(y) = D(x) + \eta(x,t)\rho(x), \quad \tau = t, \quad N(y,\tau) = \eta(x,t), \quad U(y,\tau) = u(x,t)$$

The leading term is defined from linearized shallow water with 2 fixed boundaries:

$$N_\tau + (D(y)U)_y = 0, \quad U_\tau + N_y = 0, \quad y \in [a,b], \quad E^2 = \|N\|^2 + \|\sqrt{y}\,U\|^2 < \infty$$

Finally: $x = y - \varepsilon N_1 \frac{\rho(y)}{D'(y)}, \quad \eta(x,t) = N(y,t), \quad u(x,t) = U(y,t)$



3-nodes

4-nodes

Carrier-Greenspan transformation for standing waves and experimental studies

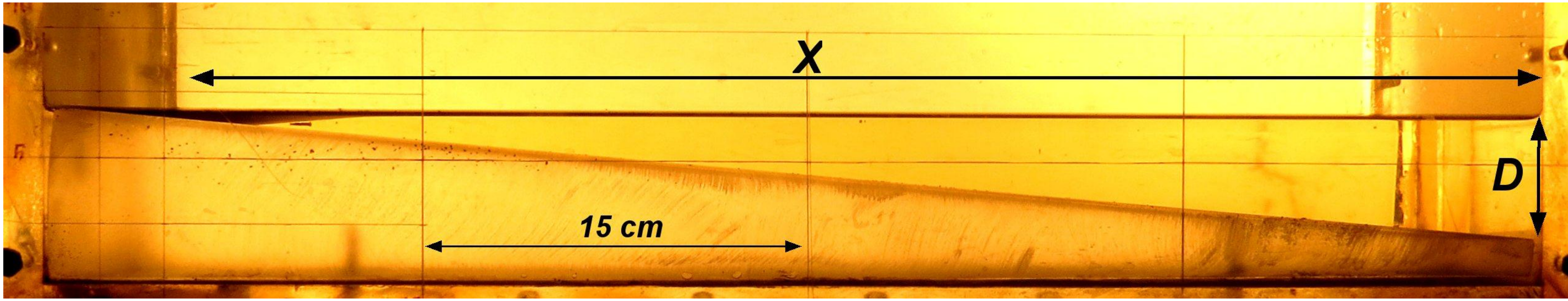
Nonlinear Shallow water equations:

$$\eta_t + \left((D(x) + \eta)u\right)_x = 0, \quad u_t + g\eta_x + uu_x = 0, \quad D = \gamma(x - x^0), \quad u|_{x=b} = 0.$$

Carrier-Greenspan transformation:

The linear equation with nonlinear boundary condition

$$N_\tau + (yU)_y = 0, \quad U_\tau + N_y = 0, \quad U(Y(\tau),\tau) = 0, \quad Y(\tau) = b + N(Y(\tau),\tau),$$
$$\|N\|^2 + \|\sqrt{y}\,U\|^2 < \infty$$



Shallow water: slopping bottom, formal asymptotics

Formal series: $$N(y,\tau,\varepsilon) = \varepsilon N_1^{w(\varepsilon)} + \varepsilon^2 N_2^{w(\varepsilon)} + \ldots,$$

Leading term: $$N_1^{w(\varepsilon)} \equiv \cos(w_0\tau)\,\mathbf{J}_0(2w(\varepsilon)\sqrt{y}) \qquad w_0 = \mu_n/2$$

$$U_1^{w(\varepsilon)} \equiv \sin(w_0\tau)\,\frac{1}{\sqrt{y}}\,\mathbf{J}_1(2w(\varepsilon)\sqrt{y})$$

Corrected frequency to avoid resonances: $$w(\varepsilon) = w_0 + \varepsilon^2 w_2 + \ldots$$

Boundary condition: $$U_1(b,\tau) = 0$$

$$U_2(b,\tau) = -U_{1y}(b,\tau)N_1(b,\tau) = \xi_1 \sin(2w_0\tau)$$

First correction: $$N_2 = c_2\cos(2w\tau)\,\mathbf{J}_0(4w\sqrt{y}) \qquad c_2 = -\tfrac{w_0}{2}\mathbf{J}_0(2w_0)\,\mathbf{J}_1'(2w_0)\,\left(\mathbf{J}_1(4w_0)\right)^{-1}$$

Boundary for U_3: $$U_3^{w_0}(b,\tau) = \xi_2\sin(3w_0\tau) + \xi_3\sin(w_0\tau) - 2w_2\mathbf{J}_1'(2w_0)\sin(w_0\tau)$$

defines phase shift w_2: $$w_2 = \xi_3/2\mathbf{J}_1'(2w_0)$$

Second correction: $$N_3 = c_3\cos(3w_0\tau)\,\mathbf{J}_0(6w_0\sqrt{y}) \qquad c_3 = \frac{\xi_2}{\mathbf{J}_1(6w_0)}$$

Etc...

Shallow water: slopping bottom, the leading term

Formal series: $N(y,\tau,\varepsilon) = \varepsilon N_1^{w(\varepsilon)} + \varepsilon^2 N_2^{w(\varepsilon)} + \ldots,$

The Leading term for linearized system:

$$N_1^{w(\varepsilon)} \equiv \cos(w_0\tau)\,\mathbf{J}_0(2w(\varepsilon)\sqrt{y}) \qquad w_0 = \mu_n/2$$

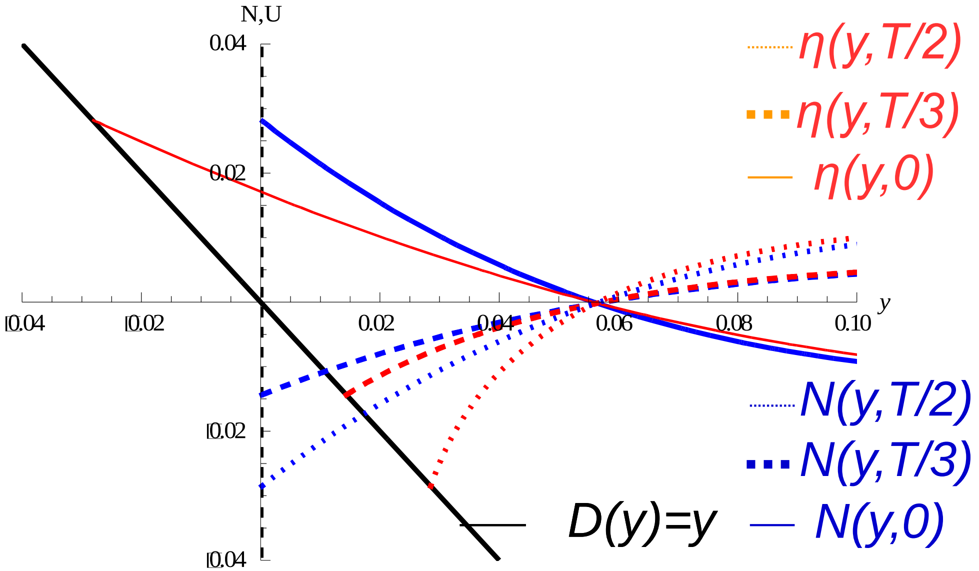
$$U_1^{w(\varepsilon)} \equiv \sin(w_0\tau)\,\frac{1}{\sqrt{y}}\,\mathbf{J}_1(2w(\varepsilon)\sqrt{y})$$

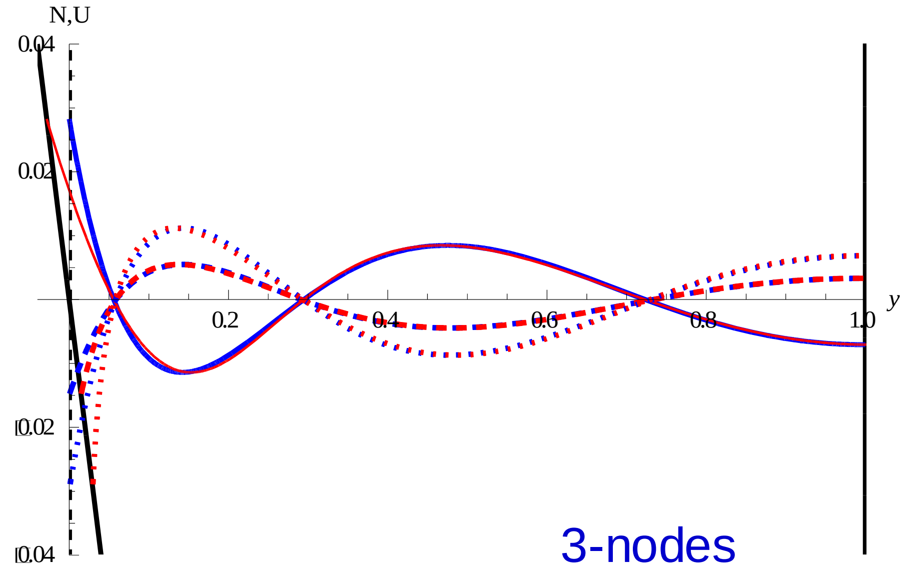
Reduced C—G transform: substitute $\tau(t,y,\varepsilon) = t + O(\varepsilon)$ into *N(y,τ), U(y,τ)*

And get the Leading term for shallow water – parametrically defined via *y ϵ [0,1]*:

$$x = y - \varepsilon N_1^{w_0}(y,t) + O(\varepsilon^2)$$

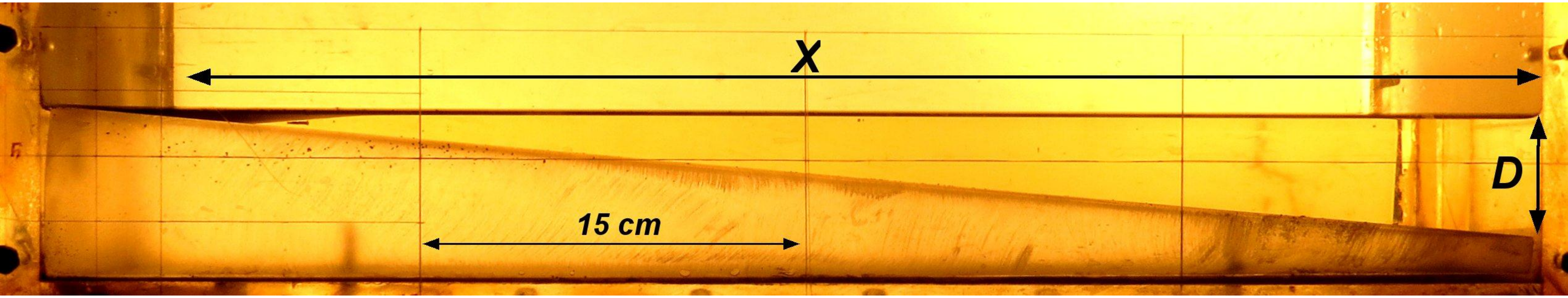
$$\eta(x,t) = \varepsilon N_1^{w_0}(y,t) + O(\varepsilon^2), \quad u(x,t) = \varepsilon U_1^{w_0}(y,t) + O(\varepsilon^2)$$

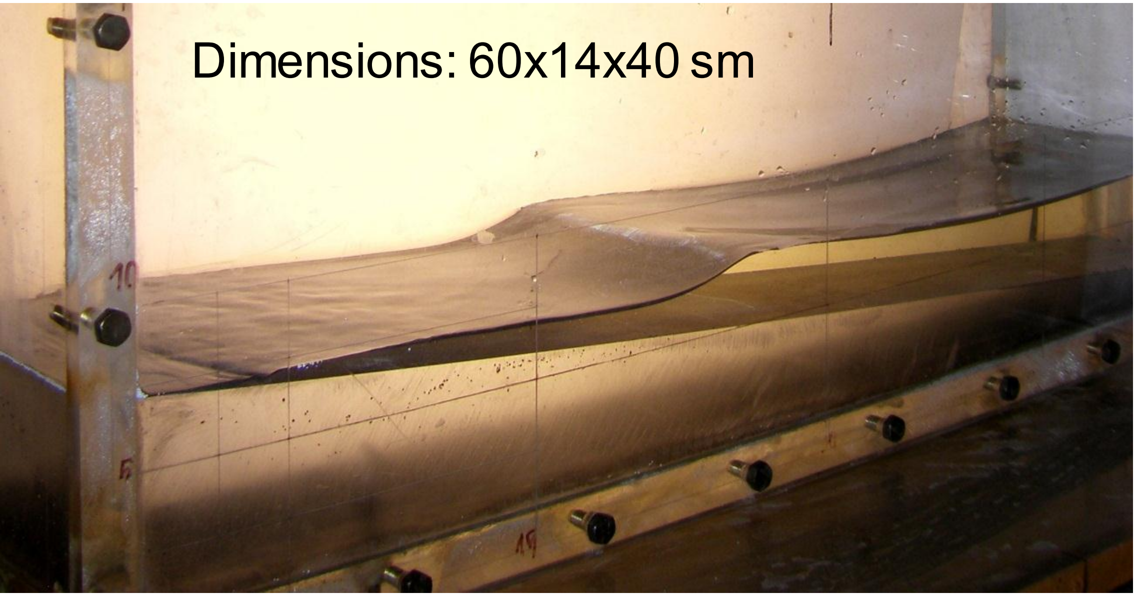




3-nodes

Experimental setup: parametric resonance





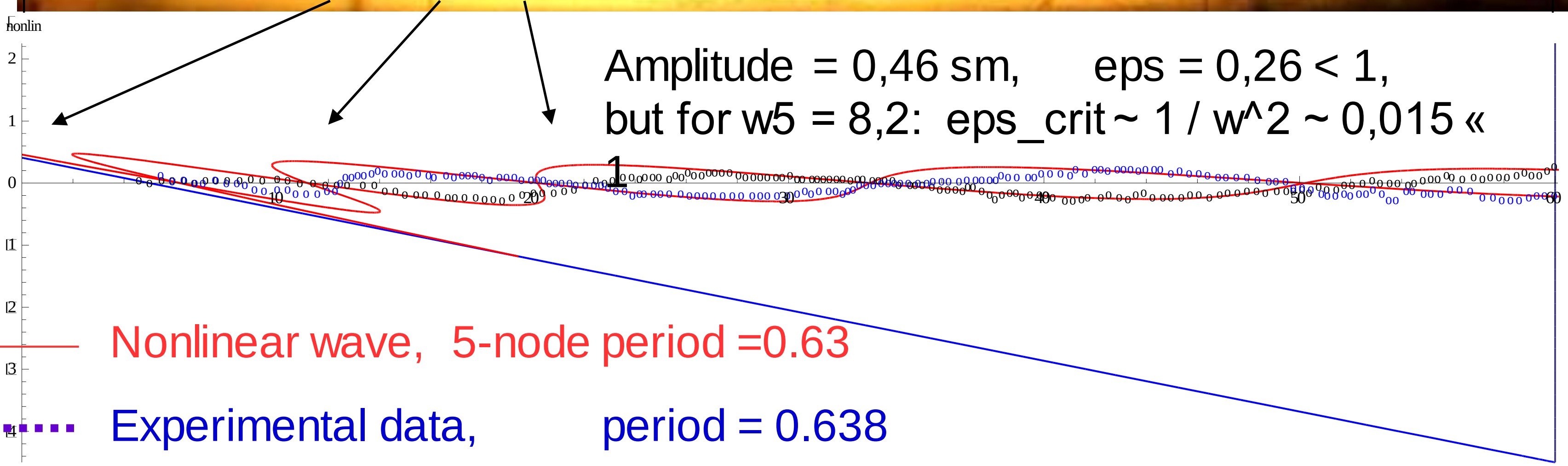
Gravity waves on the free surface
in rectangular vessel
(length = 60sm, width = 14 sm)
with slopping bottom (D:X = 4,5sm : 55sm)

Surface waves are induced
by vertical oscillations of vessel
with parametric resonance
(Oscillations period = waves period / 2)

Video 30 and 120 frames per sec,
editing in ImageJ

Experiment: comparison

Break of nonlinear wave, *J*=0



Amplitude = 0,46 sm, eps = 0,26 < 1,
but for w5 = 8,2: eps_crit ~ 1 / w^2 ~ 0,015 « 1

Nonlinear wave, 5-node period =0.63

Experimental data, period = 0.638

Спасибо за внимание!

Будьте здоровы!