Higher-Dimensional Generalizations of the Chiral Field Equations

V. E. Zakharov

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Introduction

Two-dimensional integrable relativistic-invariant systems are classical objects of the theory of integrable systems. The first $2+1$ integrable system was offered by Manakov and Zakharov in 1981. The authors were sure that this system is relativistically invariant but this was not true. The system is only "semi-invariant" or relativistically invariant in the linear approximation. This was discovered by R.S. Ward in 1988 who offered another $2+1$ generalisation of the chiral field equation. During next ten years several other articles on this subject were published then interest to this topic faded. In this talk we consider the generalized Manakov-Zakharov system, discuss three and four dimensional generalisations of the chiral field equations, and explain methods for construction of its exact solutions.

General M–Z system

Let Z_1, Z_2, Z_3, Z_4 be complex variables and $Y(Z_1, \ldots Z_4)$ be $N \times N$ complex valued invertible matrix satisfying to equation

$$\frac{\partial}{\partial Z_1}\left(\frac{\partial}{\partial Z_4}Y \cdot Y^{-1}\right) - \frac{\partial}{\partial Z_3}\left(\frac{\partial}{\partial Z_2}Y \cdot Y^{-1}\right) = 0 \tag{1}$$

This system admits the following Lax representation

$$\left(\lambda\frac{\partial}{\partial Z_1} + \frac{\partial}{\partial Z_2}\right)\Psi + A\Psi = 0, \quad \left(\lambda\frac{\partial}{\partial Z_3} + \frac{\partial}{\partial Z_4}\right)\Psi + B\Psi = 0 \tag{2}$$

Here $\lambda \in \mathbb{C}$ is the spectral parameter.

This system is accomplished by the asymptotic relation

$$\Psi \to I + \frac{P}{\lambda} + \ldots \quad \text{at} \quad \lambda \to \infty$$

$Y = \Psi|_{\lambda=0}$ is the solution of system (1) which we will call the generalized M–Z system. To prove this fact we let λ approach zero. Then

$$A = -\frac{\partial Y}{\partial Z_2} Y^{-1}, \quad B = -\frac{\partial Y}{\partial Z_4} Y^{-1}$$

If we let λ approach infinity we find

$$A - \frac{\partial P}{\partial Z_1} = 0, \quad B - \frac{\partial P}{\partial Z_3} = 0 \quad \text{here} \quad \frac{\partial A}{\partial Z_3} = \frac{\partial A}{\partial Z_1}$$

By combining these expressions we obtain the general Z–M system. Notice that in the special case $Z_3 = -Z_4 \;\; Z_2 = Z_1$ this system takes the symmetric form

$$\frac{\partial}{\partial Z_1}\left(\frac{\partial}{\partial Z_3} Y \cdot Y^{-1}\right) + \frac{\partial}{\partial Z_3}\left(\frac{\partial}{\partial Z_1} Y \cdot Y^{-1}\right) = 0$$

The Lax pair (2) turns to the form

$$\frac{\partial \Psi}{\partial Z_1} + \frac{A}{\lambda + 1}\Psi = 0, \quad \frac{\partial \Psi}{\partial Z_3} + \frac{B}{\lambda - 1}\Psi = 0$$

The symmetric equation is a Lagrangian system. The Lagrangian is

$$L = \mathrm{tr}\left(\frac{\partial Y}{\partial Z_1} Y^{-1} \frac{\partial Y}{\partial Z_3} Y^{-1}\right)$$

Thereafter we will call Ψ and Ψ^{-1} the wave function and the inverse wave function. The simplest example of wave functions are

$$\Psi = I - \frac{\lambda_0 - \mu_0}{\lambda - \lambda_0} P, \quad \Psi^{-1} = I - \frac{\lambda_0 - \mu_0}{\lambda - \mu_0} P$$

where $P = P^2$ is the projective operator.

Thereafter we consider only the simplest case when rank of P is 1. In this case P is bivector

$$P = \frac{|p\rangle\,\langle q|}{\langle q|p\rangle}, \qquad P^2 = P$$

Here $|p\rangle$ is a vector, $\langle q|$ is a covector. The Lax pair can be rewritten as follows

$$A = -\left(\lambda\frac{\partial}{\partial Z_1} + \frac{\partial}{\partial Z_2}\right)\Psi\cdot\Psi^{-1}, \quad B = -\left(\lambda\frac{\partial}{\partial Z_3} + \frac{\partial}{\partial Z_4}\right)\Psi\cdot\Psi^{-1}$$

A, B are rational functions with simple poles at $\lambda = \lambda_0$, $\lambda = \mu_0$. Hence we must demand that residues at that poles are zero and rewrite the potentials as follows

$$A, B = -\frac{\lambda_0 - \mu_0}{\lambda - \lambda_0} D_\lambda\, P\left(I - \frac{\lambda_0 - \mu_0}{\lambda - \mu_0} P\right)$$

Here $D_\lambda = \lambda\frac{\partial}{\partial Z_1} + \frac{\partial}{\partial Z_2}, \lambda\frac{\partial}{\partial Z_3} + \frac{\partial}{\partial Z_4}$ and $D_\lambda\, P = D_\lambda\,|p\rangle\,\langle q| + |p\rangle\,D_\lambda\,\langle q|$

Now let λ approach to λ_0 then

$$I - \frac{\lambda_0 - \mu_0}{\lambda - \mu_0} P \to I - P$$

Note that $\hat{P} = 1 - P$ is also a projective operator: $\hat{P}^2 = \hat{P}$. Moreover

$$P\hat{P} = \hat{P}P = 0 \quad \text{and} \quad \langle q|\hat{P}| = 0$$

Thereafter we will consider that Ψ is 2×2 matrix. It means that $\hat{P}$ is a bivector like P. Thus

$$\hat{P} = \frac{|f\rangle \langle g|}{\langle g|f\rangle} \qquad \langle g|p\rangle = 0 \qquad \langle q|f\rangle = 0$$

Potentials A, B are rational functions with two poles at $\lambda = \lambda_0$ and $\lambda = \mu_0$ and do not depend on λ. Thus residues at both poles must be zero.

Cancelling of the residue at $\lambda = \lambda_0$ leads to equation

$$D_{\lambda_0} P(1-P) = 0$$

Cancelling of the residue at $\lambda = \mu_0$ gives

$$-D_{\mu_0} P \cdot P = D_{\mu_0} (1-P) \cdot P = 0$$

Here

$$D_{\lambda_0} = \begin{cases} \lambda_0 \frac{\partial}{\partial_{Z_1}} + \frac{\partial}{\partial_{Z_2}} \\ \lambda_0 \frac{\partial}{\partial_{Z_3}} + \frac{\partial}{\partial_{Z_4}} \end{cases} \quad D_{\mu_0} = \begin{cases} \mu_0 \frac{\partial}{\partial_{Z_1}} + \frac{\partial}{\partial_{Z_2}} \\ \mu_0 \frac{\partial}{\partial_{Z_3}} + \frac{\partial}{\partial_{Z_4}} \end{cases}$$

Later one finds

$$|D_{\mu_0} f\rangle = 0 \qquad \langle D_{\lambda_0} q| = 0$$

These equations can be resolved as follows. Let

$$\Psi_0 = T\left(Z_1 + \lambda Z_2, Z_3 + \lambda Z_4\right)$$

be a general solution to the system

$$\left(\lambda\frac{\partial}{\partial Z_1} + \frac{\partial}{\partial Z_2}\right)\Psi_0 = 0 \qquad \left(\lambda\frac{\partial}{\partial Z_3} + \frac{\partial}{\partial Z_4}\right)\Psi_0 = 0$$

Let

$$F = T(Z + \lambda_0 Z_2, Z_3 + \lambda_0 Z_4) \qquad G = T^{-1}(Z + \mu_0 Z_2, Z_3 + \mu_0 Z_4)$$

Then equations can be resolved as follows

$$|f\rangle = G\,|f_0\rangle \quad \langle q| = \langle q_0|\,F$$

If vector $|f\rangle$ and covector $\langle q|$ are known then the reconstruction of $|g\rangle$ and $\langle p|$ is a pure algebraic problem which we discuss later. As far as P is known, the solution Y is

$$Y = 1 - \frac{\lambda_0 - \mu_0}{\mu_0} P \qquad Y^{-1} = 1 + \frac{\lambda_0 - \mu_0}{\mu_0} P$$

The trick that we used for construction of exact solution of nonlinear system (1) is the simplest example of the "dressing method". That was "dressing on the trivial background". To construct a more general class of solutions we will start with arbitrary chosen solution of Lax system. We denote it $\Psi(\lambda, Z_1, Z_2, Z_3, Z_4)$.

$$Y = \Psi(0, Z_1, Z_2, Z_3, Z_4)$$

Now we denote

$$F = \Psi(\lambda, Z_1, Z_2, Z_3, Z_4) \qquad G = \Psi^{-1}(\mu, Z_1, Z_2, Z_3, Z_4)$$

We will seek a new solution $\hat{\Psi}$ of Lax system as follows

$$\hat{\Psi} = \left(1 + \frac{\lambda_0 - \mu_0}{\lambda - \lambda_0}\hat{P}\right)\Psi \qquad \hat{\Psi}^{-1} = \Psi\left(1 - \frac{\lambda_0 - \mu_0}{\lambda - \mu_0}\hat{P}\right)$$

Here λ_0, μ_0 are arbitrary complex numbers, and

$$\hat{P} = \hat{P}^2 \quad \text{and} \quad \tilde{P} = \tilde{P}^2 \quad \tilde{P} = 1 - \hat{P}$$

are complimentary projectors. In the case $N = 2$ they are bivectors

$$\hat{P} = \frac{|\hat{p}\rangle\,\langle\hat{q}|}{\langle\hat{p}|\hat{q}\rangle} \qquad 1 - \hat{P} = \frac{|\hat{f}\rangle\,\langle\hat{g}|}{\langle\hat{f}|\hat{g}\rangle}$$

New vector $|\hat{p}\rangle$ and covector $\langle\hat{q}|$ are given by expressions

$$|\hat{f}\rangle = G|\hat{f}_0\rangle \qquad \langle\hat{q}| = \langle\hat{q}_0|f|$$

The new solution of system (1) is given by expressions

$$\hat{Y} = \left(1 - \frac{\lambda_0 - \mu_0}{\lambda_0}\hat{P}\right) Y \qquad \hat{Y}^{-1} = Y^{-1}\left(1 + \frac{\lambda_0 - \mu_0}{\mu_0}\right) P$$

The constructed solution is called one-soliton, but this is not exact term. We offer expression "the one-pole solution". Constructing of one pole solution gives way for construction of much more general "n-pole solution".

Again we can use the dressing method. Suppose we know one solution of (2) with wave function $\Psi(\lambda, Z_1 \dots Z_4)$. One can seek for the dressed n-pole solution in the form

$$\tilde{\Psi} = \chi(\lambda)\Psi(\lambda)$$

Here $\chi(\lambda)$ is rational function with simple poles. One can seek it in the form

$$\chi = I + \sum_{k=1}^{n} \frac{R_k}{\lambda - \lambda_k}$$

The inverse function is also rational

$$\chi^{-1} = I + \sum_{k=1}^{n} \frac{S_k}{\lambda - \mu_k}$$

Here λ_k, μ_k are some complex numbers. We consider only the case when R_k S_k have rank 1. In this case they can be presented as tensor products

$$R_k = |p_k\rangle \langle q_k| \qquad S_k = |f_k\rangle \langle g_k|$$

Vector $|f_k\rangle$ and covector $\langle g_k|$ can be found using the dressing function Ψ_0, obeying the following equations

$$|f_k\rangle = F_k|f_{0\,k}\rangle \qquad \langle q_k| = \langle q_{0\,k}|f_k \qquad F_k = \Psi_0(\lambda_k) \qquad G_k = \Psi_0^{-1}(\mu_k)$$

Again $|f_{0\,k}\rangle$ and $\langle q_{0\,k}|$ are arbitrary constant vectors and covectors.

Vectors $p_k\rangle$ and covectors $\langle g_k|$ can be found by solution of following systems of linear algebraic equations

$$|f_l\rangle + \sum_{k=1}^{n} \frac{|p_k\rangle \langle q_k|f_l\rangle}{\mu_l - \lambda_k} = 0 \qquad |q_l\rangle + \sum_{k=1}^{n} \frac{|g_l\rangle \langle f_k|q_l\rangle}{\lambda_l - \mu_k} = 0$$

If all λ_l and μ_k are different and denominators are not zero, this system has an unique solution. Special cases when some λ_k and μ_k coincide can be studied by the limiting transition from the generic case.

Notice that expressions for χ and χ^{-1} do not change if ordering of poles $\lambda_1, \ldots, \lambda_k$ and $\mu_1, \ldots, \mu_k$ is different.

There is another way to construct n-poles solution. Suppose we define some ordering of poles and will seek the wave function as a product of one-pole solution

$$\chi = \prod_{k=1}^{n} \left(I + \frac{\lambda_k - \mu_k}{\lambda - \lambda_k} P_k \right)$$

$$\chi^{-1} = \prod_{k=1}^{n} \left(I - \frac{\lambda_{n-k+1} - \mu_{n-k+1}}{\lambda - \mu_{n-k+1}} P_{n-k+1} \right)$$

Here $P_k = P_k^2$ are some projective operators. All of them can be found by implementation of consequence dressing, which is a simple generalization of the procedure used for construction of a general one-pole solution discussed above.

Both methods for construction of n-poles solution give identical results.

Generalized M–Z system

Let us suppose that in (1) $Z_4 = \bar{Z}_1 = u$ $Z_3 = \bar{Z}_2 = v$. Then we obtain the system

$$\frac{\partial}{\partial u}\left(\frac{\partial Y}{\partial \bar{u}}Y^{-1}\right) - \frac{\partial}{\partial v}\left(\frac{\partial Y}{\partial \bar{v}}Y^{-1}\right) = 0 \qquad (3)$$

In the particular case when $\bar{u} = u = \tau$ this is Manakov–Zakharov (MZ) hyperbolic system

$$\frac{\partial}{\partial \tau}\left(\frac{\partial Y}{\partial \tau}Y^{-1}\right) = \frac{\partial}{\partial v}\left(\frac{\partial Y}{\partial \bar{v}}Y^{-1}\right) \qquad (4)$$

We will study the general system (3), having in the mind that the transition to the special case (4) can be performed easily.

The Lax pair looks now as follows

$$\left(\lambda\frac{\partial}{\partial u} + \frac{\partial}{\partial \bar{v}}\right)\Psi + A\Psi = 0 \qquad \left(\lambda\frac{\partial}{\partial v} + \frac{\partial}{\partial \bar{u}}\right)\Psi + B\Psi = 0$$

So far Y is a matrix function free of any limitation. One-pole solution is given by the construction described earlier. The dressing function T is still the function of two variables

$$T = T\left(\bar{v} + \frac{1}{\lambda}u, \bar{u} + \frac{1}{\lambda}v\right)$$

This system is too general to be really interesting. Now we attract attention to one fact. If Y is the solution of (3) then $\pm Y^{+}$ also are solutions of this system. Thereafter one can assume that $Y^{+} = Y$. Imposing of this reduction makes possible to impose a strict involution on the wave function $\Psi(\lambda)$.

Let us consider function $\hat{\Psi}(\lambda)$ defined by condition

$$\hat{\Psi}(\lambda) = \Psi^{+}\left(\frac{1}{\lambda}\right)$$

Here $\Psi^{+}(\lambda) = \Psi^{+}(\bar{\lambda})$. We claim that $\hat{\Psi}$ satisfies the condition

$$\hat{\Psi}(\lambda) = \Psi^{-1}(\lambda)Y, \quad Y = \Psi(0)$$

To prove this fact we plug $\hat{\Psi}(\lambda)$ into Lax system and find that $\hat{\Psi}$ satisfies equations

$$\left(\lambda\frac{\partial}{\partial v}+\frac{\partial}{\partial \bar{u}}\right)\hat{\Psi}+\lambda\hat{\Psi}A^{+}=0 \qquad \left(\lambda\frac{\partial}{\partial u}+\frac{\partial}{\partial \bar{v}}\right)\hat{\Psi}+\lambda B^{+}\hat{\Psi}^{+}=0$$

At this time the inverse matrix satisfies to system of equations

$$\left(\lambda\frac{\partial}{\partial v}+\frac{\partial}{\partial \bar{u}}\right)\Psi^{-1}=\Psi^{-1}B \qquad \left(\lambda\frac{\partial}{\partial u}+\frac{\partial}{\partial \bar{v}}\right)\Psi^{-1}=\Psi^{-1}A$$

After cancelling by $\Psi^{-1}(\lambda)$ we end up with the following relation

$$BY+\left(\lambda\frac{\partial}{\partial v}+\frac{\partial}{\partial \bar{u}}\right)Y+\lambda YA^{+}=0\Psi^{-1}=\Psi^{-1}B$$

$$AY+\left(\lambda\frac{\partial}{\partial u}+\frac{\partial}{\partial \bar{v}}\right)Y+\lambda YB^{+}=0$$

Separating the constant and linear by λ terms we obtain

$$\frac{\partial Y}{\partial \bar{u}} + BY = 0 \qquad \frac{\partial Y}{\partial \bar{v}} + AY = 0$$

or

$$B = -\frac{\partial Y}{\partial \bar{u}} Y^{-1} \qquad A = -\frac{\partial Y}{\partial \bar{v}} Y^{-1}$$

and

$$\frac{\partial Y}{\partial v} + YA^{+} = 0 \qquad \frac{\partial Y}{\partial u} + YB^{+} = 0$$

After applying to this system the operation of conjugation and using relation $Y^{+} = Y$ we return to equations which are satisfied.

The considered involution implies a strong restriction on position of poles of direct and inverse wave functions Ψ, Ψ^{-1}. Suppose that both Ψ and Ψ^{-1} are rational functions

$$\Psi(\lambda) = I + \sum_{k=1}^{n} \frac{R_k}{\lambda - \lambda_k} \qquad \Psi^{-1}(\lambda) = I - \sum_{k=1}^{n} \frac{S_k}{\lambda - \mu_k}$$

Let us calculate $\hat{\Psi}(\lambda)$

$$\hat{\Psi}(\lambda) = I + \sum_{k=1}^{n} \frac{R_k^+}{\frac{1}{\lambda} - \bar{\lambda}_k}$$

This function can be expanded to partial fractions. After simple calculation we end up with the following result

$$\hat{\Psi}(\lambda) = Y^+ - \sum_{k=1}^{n} \frac{R_k^+}{\lambda - \frac{1}{\bar{\lambda}_0}} \frac{1}{\bar{\lambda}_0^2}, \qquad Y^+ = I - \sum_{k=1}^{n} \frac{R_k^+}{\bar{\lambda}_k} = Y$$

Then we find that

$$\mu_k = \frac{1}{\bar{\lambda}_k} \qquad S_k = R_k^+ \frac{1}{\bar{\lambda}_k^2} Y^{-1}$$

Let us consider the simplest case $n = 1$. Now we have only one pair of poles λ_1 and $\mu_1 = \frac{1}{\bar{\lambda}_1}$. Then

$$\Psi = I + \frac{\lambda_1 - \frac{1}{\bar{\lambda}_1}}{\lambda - \lambda_1} P \qquad \Psi^{-1} = I - \frac{\lambda_1 - \frac{1}{\bar{\lambda}_1}}{\lambda - \frac{1}{\bar{\lambda}_1}} P$$

Substitution leads to the relation $P^+ = P$. It means that

$$P = \frac{|\bar{q}\rangle \langle q|}{\langle q|\bar{q}\rangle}$$

Here

$$\langle q| = \langle q_0|G \qquad G = T^{-1} \left(\bar{v} + \bar{\lambda}_1 u + \bar{u} + \frac{1}{\bar{\lambda}_1} \bar{v} \right)$$

T is an arbitrary function of one complex variable.

If reduction is satisfied, all projectors are Hermitian

$$P_i^T = P_i \quad \text{and} \quad \mu_k = \frac{1}{\lambda_k}$$

Generalized Ward's system

Let us suppose that all Z_i are real. We rename them to x_i. Equation (1) reads now

$$\frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial x_4}Y \cdot Y^{-1}\right) - \frac{\partial}{\partial x_3}\left(\frac{\partial}{\partial x_2}Y \cdot Y^{-1}\right) = 0 \qquad (5)$$

The particular case of this system, when

$$x_1 = x_4 = x \quad x_2 = \frac{1}{2}(t+y) \quad x_3 = \frac{1}{2}(t-y)$$

was studied by R. S. Ward. Hence we call equation (5) the generalized Ward's system. We notice that Y in (5) can be real or complex valued matrix function. We will discuss the general case when Y is complex-valued. The reduction to the real case can be done easily.

The Lax pair (2) looks now as follows

$$\left(\lambda\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right)\Psi + A\Psi = 0, \qquad \left(\lambda\frac{\partial}{\partial x_3}+\frac{\partial}{\partial x_4}\right)\Psi + B\Psi = 0$$

Suppose that A, B are anti-Hermitian

$$A^{+} = -A \quad B^{+} = -B$$

Now system takes form

$$\left(\lambda\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right)\hat{\Psi} - A\hat{\Psi} = 0 \qquad \left(\lambda\frac{\partial}{\partial x_3}+\frac{\partial}{\partial x_4}\right)\hat{\Psi} - B\hat{\Psi} = 0$$

The inverse wave function Ψ^{-1} satisfies to exactly the same equation. It does not mean that $\hat{\Psi}$ and Ψ^{-1} coincide.

It means that there is a relation

$$\Psi^{+}(\bar{\lambda}) = \Psi^{-1}(\lambda)R(\lambda)$$

where $R(\lambda)$ satisfies to equations

$$\left(\lambda\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)R - [A\,R] = 0 \qquad \left(\lambda\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}\right)R - [B\,R] = 0$$

For solution of system (5) one obtains reduction

$$Y^{+} = Y^{-1}R \quad R = R(0)$$

if $R = 1$, Y is a unitary matrix. *Warning!* This function does not necessary belong to SU_N group because $h = \det Y$ in a general case is not unit. It obeys the condition

$$|h|^2 = 1$$

It should be mentioned that most of exact solutions described in Ward's articles do not satisfy the condition $h = 1$, hence do not belong to SU_N.

We see that the class of described involution is very broad. However thereafter we will put $R = 1$ such that Y is an unitary matrix but condition $h = 1$ we consider as odd and not necessary. Thereafter we will consider that

$$\Psi^+(\bar{\lambda}) = \Psi^{-1}(\lambda)$$

Let Ψ and Ψ^{-1} be presented by expansions to partial fractions. Substituting these relations gives $\mu_k = \bar{\lambda}_k \; S_k = R_k^+$.

Again, let us consider the one-pole solution

$$\Psi_0 = I + \frac{\lambda_0 - \bar{\lambda}_0}{\lambda - \lambda_0} P \qquad \Psi^{-1} = I - \frac{\lambda_0 - \bar{\lambda}_0}{\lambda - \bar{\lambda}_0} P$$

The above involution means that $P^+ = P$, hence P is a Hermitian projector.

In the same way in the product of one-pole solutions

$$\Psi = \prod \left(I - \frac{\lambda_k - \bar{\lambda}_k}{\lambda - \bar{\lambda}_k} P_k \right)$$

all "partial" projectors are Hermitian $P_k^+ = P_k$.

Notice that for one-pole solution

$$Y = I - \frac{\lambda_0 - \bar{\lambda}_0}{\lambda_0} P \qquad \det Y = \frac{\bar{\lambda}_0}{\lambda_0} \neq 1$$

Hence this solution does not belong to SU_N. To construct the solution belonging to SU_N one should consider the two-pole solutions $\lambda_1 = \lambda_0$, $\lambda_2 = \bar{\lambda}_0$. In this case poles of direct and inverse wave function coincide. A detailed study of this system is an interesting problem but this is beyond the scope of this article.

3+1 integrable systems

Let us consider equation (1) after the following simplifications

$$\lambda_1 = \xi \quad x_1 = \eta \quad x_2 = u \quad x_3 = t\xi = \frac{1}{2}(t+z) \quad x_4 = \frac{1}{2}(t-z)$$

$$\frac{\partial}{\partial \xi}\left(\frac{\partial}{\partial \eta} Y \cdot Y^{-1}\right) - \frac{\partial}{\partial \bar{u}}\left(\frac{\partial}{\partial u} Y \cdot Y^{-1}\right)$$

Now we introduce the Lax pair

$$\left(\lambda \frac{\partial}{\partial \xi} + \frac{\partial}{\partial u}\right)\Psi + A\Psi = 0 \qquad \left(\lambda \frac{\partial}{\partial \eta} + \frac{\partial}{\partial \bar{u}}\right)\Psi + B\Psi = 0$$

Here $Y = \Psi(0)$ and

$$A = -\frac{\partial Y}{\partial u}\Psi^{-1} \quad B = -\frac{\partial Y}{\partial \bar{u}}\Psi^{-1}$$

The general solution of Lax system is

$$\Psi = \Psi(\xi + \lambda u, \eta + \lambda \bar{u})$$

Here Ψ is an arbitrary matrix-valued function of two complex variables.

We will consider one important class of special solutions, when Ψ is a function of one only complex variable

$$\Psi = \Psi\left(\xi + c\eta + \lambda(u + c\bar{u})\right)$$

where c is real constant. Now

$$\xi + c\eta = \frac{1}{2}\left[(1+c)t + (1-c)Z\right]$$

If $c = -1$, this is a stationary solution. If $c = 1$, this is a solution homogeneous along Z axis.

In a general case $c \neq \pm 1$, this solution describes waves propagating along Z-axis. Now

$$\xi + c\eta = \frac{1}{2}(1-c)(Z+vt) \qquad v = \frac{1+c}{1-c}$$

Thereafter we will consider only the case when Ψ is a diagonal matrix. Now

$$F = \begin{Vmatrix} F_1 & 0 \\ 0 & F_2 \end{Vmatrix} \qquad G = \begin{Vmatrix} \frac{1}{\tilde{F}_1} & 0 \\ 0 & \frac{1}{\tilde{F}_2} \end{Vmatrix}$$

$$\begin{aligned} F_1 &= F_1\left(\xi + c\eta + \lambda_0(u + c\bar{u})\right) & F_2 &= F_2\left(\xi + c\eta + \lambda_0(u + c\bar{u})\right) \\ \tilde{F}_1 &= F_1\left(\xi + c\eta + \mu_0(u + c\bar{u})\right) & \tilde{F}_2 &= F_2\left(\xi + c\eta + \mu_0(u + c\bar{u})\right) \end{aligned}$$

Then we define the initial vector and initial covector

$$|p_0\rangle = \begin{vmatrix} p_1 \\ p_2 \end{vmatrix} \qquad \langle q_0| = (q_1, q_2)$$

After simple intermediate calculations, the components of the solution Y are the following:

$$Y_{11} = 1 - \frac{\lambda_0 - \mu_0}{\lambda_0} \frac{p_1 q_1}{\Delta} F_1 \tilde{F}_2 \qquad Y_{12} = -\frac{\lambda_0 - \mu_0}{\lambda_0} \frac{p_1 q_2}{\Delta} F_1 \tilde{F}_1$$

$$Y_{21} = -\frac{\lambda_0 - \mu_0}{\lambda_0} \frac{p_2 q_1}{\Delta} F_2 \tilde{F}_2 \qquad Y_{22} = 1 - \frac{\lambda_0 - \mu_0}{\lambda_0} \frac{p_2 q_2}{\Delta} p_2 q_2 F_2 \tilde{F}_1$$

$$\Delta = p_1 q_1 F_1 \tilde{F}_2 + p_2 q_2 F_2 \tilde{F}_1$$

One-soliton solution

Let us put

$$F_1 = e^{a(\xi+c\eta+\lambda_0(u+c\bar{u}))} \qquad F_2 = e^{-a(\xi+c\eta+\lambda_0(u+c\bar{u}))}$$

$$\tilde{F}_1 = e^{a(\xi+c\eta+\mu_0(u+c\bar{u}))} \qquad \tilde{F}_2 = e^{-a(\xi+c\eta+\mu_0(u+c\bar{u}))}$$

Now suppose that a, λ_0, μ_0 are pure imaginary

$$a = is \quad \lambda_0 = iA \quad \mu_0 = -iA$$

Then

$$F_1 = e^{is(\xi+c\eta)-As(u+c\bar{u})} \qquad F_2 = e^{-is(\xi+c\eta)-As(u+c\bar{u})}$$
$$\tilde{F}_1 = e^{is(\xi+c\eta)+As(u+c\bar{u})} \qquad \tilde{F}_2 = e^{-is(\xi+c\eta)-As(u+c\bar{u})}$$

and

$$F_1\tilde{F}_1 = e^{2is(\xi+c\eta)} \qquad F_1\tilde{F}_2 = e^{-2As(u+c\bar{u})}$$

$$\tilde{F}_1 F_2 = e^{2As(u+c\bar{u})} \qquad F_2 \tilde{F}_2 = e^{-2is(\xi+c\eta)}$$

$$\frac{\lambda_0 - \mu_0}{\lambda_0} = 2$$

If we put $q_1 = \bar{p}_1$, $q_1 = \bar{p}_2$, then

$$\Delta = |p_1|^2 e^{-2As(u+c\bar{u})} - |p_2|^2 e^{2As(u+c\bar{u})}$$

Thus the elements of the solution Y are

$$Y_{11} = 1 - \frac{|p_1|^2}{2\Delta} e^{2As(u+c\bar{u})} \qquad Y_{12} = -\frac{p_1 \bar{p}_2}{2\Delta} e^{2is}$$

$$Y_{21} = -\frac{\bar{p}_1 p_2}{2\Delta} e^{2is} = Y_{12} \qquad Y_{22} = 1 - \frac{|p_2|2}{2\Delta} e^{-2As(u+c\bar{u})}$$

Let $A > 0$. Notice that $|u + c\bar{u}| \to \infty$ as $|u| \to \infty$ at any direction on the u, $\bar{u}$ plane. Then

$$Y_{11} \to 1 \quad Y_{11} \to \frac{1}{2} \quad Y_{12} \to 0 \quad Y_{12} \to 0 \quad \text{as} \, |u| \to \infty$$

Thus, this solution can be interpreted as a one-soliton solution. This solution is essentially two-dimensional and moves along Z-axis with a constant velocity which can be an arbitrary real number (including the limiting case $v = \pm\infty$).

On the plane u, $\bar{u}$ the off diagonal elements Y_{12},, Y_{21} decay exponentially, while the diagonal elements become constants as $|u| \to \infty$. The absolute values of elements $|Y_{ij}|$ do not depend on t and Z.

This example shows that in 3+1 case the system (1) has an extremely rich class of solitonic solutions. Their detailed description is the subject for another work.

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