Quasiclassical theory of collapse of acoustic waves in media with positive dispersion

E.A. Kuznetsov

Lebedev Physical Institute, Moscow

Landau Institute for Theoretical Physics, Chernogolovka, Russia

Space Research Institute of RAS, Moscow

Skolkovo Institute of Science and Technology, Skolkovo, Russia

The XXX-th Session of the RAS scientific council Nonlinear Dynamics", IO RAS, December 20, 2021

OUTLINE

- Collapse and the role of radiation
- Variational ansatz
- Quasi-classical collapse
- Concluding remarks

Introduction: collapses vs. solitons

- In our paper (Zakharov and K., 1986) the quasiclassical collapsing solution for the 3D NLS was constructed. Besides, the whole family of quasi-classical weak collapses was predicted. The upper boundary of this family represents the self-similar weak collapsing solution.
- Based on the quasiclassical approach analogous to the Whitham procedure, we show that the strong collapsing solution of the equations and the time-dependent ansatz within the variational method have near singularity the same self-similar asymptotics.
- Here we find the whole family of the quasiclassical collapsing self-similar substitutions with the bounds corresponding to strong collapse and self-similar solution.

General properties of the KP equation

- Next, we will use the standard form for KP

$$\frac{\partial}{\partial x}\left(u_t + 6uu_x + u_{xxx}\right) - \Delta_\perp u = 0.$$

- In the Hamiltonian form, it reads

$$u_t = \frac{\partial}{\partial x}\frac{\delta \mathcal{H}}{\delta u}$$

where

$$\mathcal{H} = \int \left[\frac{u_x^2}{2} + \frac{(\nabla_\perp w)^2}{2} - u^3\right] dr \equiv \frac{1}{2}I_1 + \frac{1}{2}I_2 - I_3,$$

with $w_x = u$. Besides $\mathcal{H}$, KP conserves $P = 1/2\int u^2 d\mathbf{r} > 0$.

General properties of the KP equation

- The KP soliton is a solution $u = u(x - Vt, \mathbf{r}_\perp)$ which represents a stationary point of the Hamiltonian $\mathcal{H}$ for fixed P: $\delta(\mathcal{H} + VP) = 0$.
- For arbitrary d the value of $\mathcal{H}$ on the soliton solution can be expressed through P:
  $\mathcal{H}_s = \frac{2d-5}{7-2d} V P_s$.
- This answer can be obtained by using scaling transformations remaining P,

$$u(x, r_\perp) \to a^{-1/2} b^{(1-d)/2} u(x/a, r_\perp/b).$$

General properties of the KP equation

- Under these transformations $\mathcal{H}$ becomes a function of the scaling parameters a and b:

$$\mathcal{H}\,(a,b) = \frac{1}{2}a^{-2}I_1 + \frac{1}{2}a^2 b^{-2} I_2 - a^{-1/2} b^{(1-d)/2} I_3 .$$

- At $d = 3$ the function $\mathcal{H}(a,b)$ is unbounded from below. It follows if one considers the line $b = a^2$ which corresponds to self-similar behavior.
  The unboundedness of the Hamiltonian represents on the key criteria for the wave collapse in 3D KP.

Role of radiation

- Under scaling transformation of $\mathcal{H}$ along the parabolas $b \propto a^2$ the exponents of the quadratic terms (equal to -2) and in the cubic term ($-5/2$) do not coincide, and hence the possible collapse is not critical and should be weak, corresponds to the self-similar collapse. In such regimes the radiation of small amplitude waves from the collapsing region promotes collapse.
- In the case of the unbounded $\mathcal{H}$ from below the emission of small-amplitude waves promotes collapse. Consider a domain Ω with $\mathcal{H}_\Omega < 0$. Then, using the mean value theorem, it follows the following inequality $|u|_{\max} \geq \frac{|\mathcal{H}_\Omega|}{2P_\Omega}$.
- Hence we see that radiation promotes collapse.

Variational ansatz

- Action S for the KP equation is written as

$$S = \int \left[\frac{1}{2} w_t w_x - \frac{1}{2} w_{xx}^2 - \frac{1}{2} \left(\nabla_\perp w \right)^2 + w_x^3 \right] dt dr;$$

$\delta S = 0$ is equivalent to KP.

- Choose the test function in the form
$w = a^{1/2} b^{-1} f(\xi_\parallel, \xi_\perp), u = a^{-1/2} b^{-1} U(\xi_\parallel, \xi_\perp),$
where a and b are functions of t, $\xi_\parallel = x/a$ and $\xi_\perp = r_\perp / b$ are self-similar variables.

- Substituting test function into S and integrating over r we get the Lagrangian in terms of a and b:

$$L = a \frac{b_t}{b} M - \mathcal{H}\left(a, b\right).$$

Here M is the constant

Variational ansatz

- This gives equations for a and b

$$Ma_t = -b\frac{\partial \mathcal{H}}{\partial b}, \quad Mb_t = b\frac{\partial \mathcal{H}}{\partial a}.$$

- These equations have a stationary solution in the form of 3D solitons when $a = b = 1$. Linear stability of this solution is defined from equations for small perturbations α, β ($a = 1 + \alpha$, $b = 1 + \beta$):

$$M\alpha_t = VP\left(10\alpha - 4\beta\right), \quad M\beta_t = VP\left(19\alpha - 10\beta\right).$$

Hence, we arrive at the instability with growth rate $\gamma = \pm 2\sqrt{6}\frac{VP}{M}$.

Variational ansatz

- The nonlinear stage of this instability results in collapse because of the unboundedness of $\mathcal{H}$ as $a \to 0$ and $b \to 0$. Simple analyze the equations of motion gives the following asymptotics for a:
$a \to (t_0 - t)^{1/4}$.
- In this case, near the collapsing time $t = t_0$ the diffraction and nonlinear terms in $\mathcal{H}\,(a, b)$ are compensated each other. This regime, as we will see below, is realized for the quasi-classical initial conditions.
- Note that the trial function in the variational ansatz conserves momentum P and therefore collapse in this ansatz belongs to the strong ones.

Quasi-classical collapse

For the initial distributions for which it is possible to neglect both dispersion and diffraction at the beginning the temporal behavior of u will be defined by the Hopf equation

$$u_t + 6uu_x = 0,$$

where u depends on $r_\perp$ as a parameter. Thus, we arrive immediately to breaking where our assumption is invalid and we have to take into account both the diffraction and dispersion terms. As a result of these linear effects, the spatial oscillatory structure begins to develop.
However in 3D all types of solitons are unstable. In particular, the 1D solitons are unstable relative to the KP instability. The same statement is valid for 2D solitons (K.& Turitsyn 1982).

Quasi-classical collapse

The nonlinear stage of this instability is a collapse. It means that instead of soliton train in the 1D case one should expect the formation of the oscillatory structures containing the collapsing solitons. Thus, the solution in this case should be constructed in the form

$$u = u(r, t, \Phi(r, t))$$

where u is 2π-periodic in Φ and is slowly varying function relative to r and t. Also the slow varying functions are Φ_t and $\nabla\Phi$.

First, consider the linear case when u obeys the linear KP equation

Quasi-classical collapse

In this case, we can restrict ourselves by a one-harmonic dependence

$$u = Ae^{i\Phi} + c.c..$$

Then the first order leads us the Hamilton-Jacobi equation

$$\Phi_t + \omega(\nabla\Phi) = 0,$$

where $\omega(k) = -k_x^3 - k_\perp^2/k_x$ is the dispersion law and $k = \nabla\Phi$ is the wave vector.

The next order gives the continuity equation for

$$\frac{\partial A}{\partial t} + \frac{1}{2A}\text{div}(A^2\mathbf{v}) = 0,$$

where $\mathbf{v} = \partial\omega(\mathbf{k})/\partial\mathbf{k}$.

Quasi-classical collapse

Because of the nonlinearity of KP we have to take into account all harmonics,

$$u = \sum_{n=-\infty}^{\infty} A_n e^{in\Phi}, \quad A_n = A_{-n} = A_n^*.$$

As a result, we have for $n \neq 0$

$$\begin{aligned} \frac{\partial A_n}{\partial t} + \frac{1}{2A_n}\text{div}(A_n^2 \mathbf{v}_n) + 3\frac{\partial}{\partial x} s_n &= 0, \\ \Phi_{nt} + \omega(\nabla\Phi_n) + 3\Phi_{nx}\frac{s_n}{A_n} &= 0, \end{aligned}$$

where $\Phi_n = n\Phi$ and $s_n = \sum_{n=n_1+n_2} A_{n_1} A_{n_2}$.

Quasi-classical collapse

For zero harmonics a separate equation arises,

$$\frac{\partial A_0}{\partial t} + 3\frac{\partial}{\partial x}\langle u^2\rangle = 0.$$

Here

$$\langle u^2\rangle = \sum_{n=-\infty}^{\infty} A_n^2 = s_0.$$

It should be noted that this infinite system is overdetermined because for each $n, m \neq 0$ we get the following constraints

$$\pi_n(r,t) = \pi_m(r.t)$$

where

$\pi_n(r,t) = n^2\Phi_x^2 - \frac{s_n}{A_n}.$

Quasi-classical collapse

Let us seek for the solution of system in the self-similar form

$$A_n(r,t) = (t_0-t)^{-\alpha} f_n\left(\frac{x}{(t_0-t)^{\beta}}, \frac{r_\perp}{(t_0-t)^{\gamma}}\right),$$

$$\Phi(r,t) = \lambda^2 \int^t \frac{dt}{(t_0-t)^{\kappa+1}} + (t_0-t)^{-\kappa}\varphi\left(\frac{x}{(t_0-t)^{\beta}}, \frac{r_\perp}{(t_0-t)^{\gamma}}\right)$$

where t_0 is a collapse time, λ^2 is a constant, $\alpha, \beta, \gamma,$ and κ are unknown exponents.

After substitution we have

$$\beta = 1-\alpha, \gamma = 1-\alpha/2, \kappa = 3\alpha/2 - 1.$$

Quasi-classical collapse

The total cavity momentum P can only reduce during the collapsing time,

$$P_{cav} = \frac{1}{2}\int_{cav} u^2 d^3 r \propto (t_0 - t)^{3-4\alpha},$$

i.e. $\alpha \leq 3/4$. Hence we can see that $\alpha = 3/4$ corresponds to a strong collapse regime, coinciding with the variational anzats.

In this family the low boundary is defined from the quasi-classical criterion. It is possible to show that the quasi-classical criterion reads as $\alpha > 2/3$, it improves with time.

Quasi-classical collapse

$\alpha = 2/3$ is a low boundary of this family where the quasi-classical assumption is invalid. This value replies to the self-similar solution of the 3D KP equation describing the most rapid weak collapse.

THANKS FOR YOUR ATTENTION