The linear instability near the Akhmediev breather – the regular approach.

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Anomalous waves for the Focusing NLS

We study the anomalous waves on the focusing NLS equation (SfNLS) with periodic boundary conditions:

$$iu_t + u_{xx} + 2u^2\bar{u} = 0$$

We use the following Cauchy data (anomalous waves Cauchy problem):

$$u(x,0) = a + \epsilon v(x), \quad v(x+L) \equiv v(x), \quad |\epsilon| \ll 1,$$
$$v(x) = \sum_{j\geq 1} \left(c_j e^{ik_j x} + c_{-j} e^{-ik_j x}\right), \quad k_j = \frac{2\pi}{L} j, \quad |c_j| = O(1),$$

To simplify calculations we also assume that the period L is generic: $L \neq \pi n$, $n \in \mathbb{Z}$.

Linear perturbation theory near the background

Consider the unstable background: ($\epsilon = 0$):

$$u_0(x,t) \;=\; ae^{2i|a|^2 t}.$$

A harmonic perturbation

$$u(x,0) = a + \epsilon e^{ikx}$$

is **unstable** if $|k| < 2|a|$ and **stable** if $|k| \geq 2|a|$.

Linear stability of the background

We study periodic problem, therefore only L-periodic perturbations are considered:

$$k = k_j = \frac{2\pi}{L} j, \quad j \in \mathbb{Z}.$$

The first N harmonics are unstable, where

$$N = \left[\frac{|a|L}{\pi} \right]$$

with the growing factor in the linear mode:

$$\sigma_j = |a| k_j \sqrt{4|a|^2 - k_j^2}, \quad 1 \le j \le N,$$

All other modes are stable. They give only small corrections and we discard them.

Zero-curvature representation

Integrability of self-focusing NLS equation (SfNLS)

$$iu_t + u_{xx} + 2u^2\bar{u} = 0, \quad u = u(x,t)$$

is based on the zero-curvature representation (Zakharov-Shabat):

$$\vec{\Psi}_x(\lambda,x,t) = U(\lambda,x,t)\vec{\Psi}(\lambda,x,t), \quad \vec{\Psi}_t(\lambda,x,t) = V(\lambda,x,t)\vec{\Psi}(\lambda,x,t),$$

$$U = \begin{bmatrix} -i\lambda & iu(x,t) \\ \overline{iu(x,t)} & i\lambda \end{bmatrix},$$

$$V(\lambda,x,t) = \begin{bmatrix} -2i\lambda^2 + iu(x,t)\overline{u(x,t)} & 2i\lambda u(x,t) - u_x(x,t) \\ 2i\lambda\overline{u(x,t)} + \overline{u_x(x,t)} & 2i\lambda^2 - iu(x,t)\overline{u(x,t)} \end{bmatrix},$$

where

$$\vec{\Psi}(\lambda,x,t) = \begin{bmatrix} \Psi^1(\lambda,x,t) \\ \Psi^2(\lambda,x,t) \end{bmatrix}.$$

One unstable mode

Let us discuss the first non-trivial case $N = 1$: $\pi/|a| < L < 2\pi/|a|$.

$$u(x,0) = a\left(1 + \epsilon\left(c_1 e^{k_1 x} + c_{-1} e^{-ik_1 x}\right)\right), \quad k_1 = \frac{2\pi}{L}, \quad \epsilon \ll 1,$$

where c_1 and c_{-1} are arbitrary $O(1)$ complex parameters.

Problem: Calculate the time of the first rogue wave appearance and its position. Calculate the periodicity of appearances in terms of the Cauchy data.

The unstable mode is described by Riemann theta functions of 2 variables.

But for this special Cauchy data it admits a good approximation as a sequence of Akhmediev breathers (Grinevich–Santini).

Akhmediev breathers:

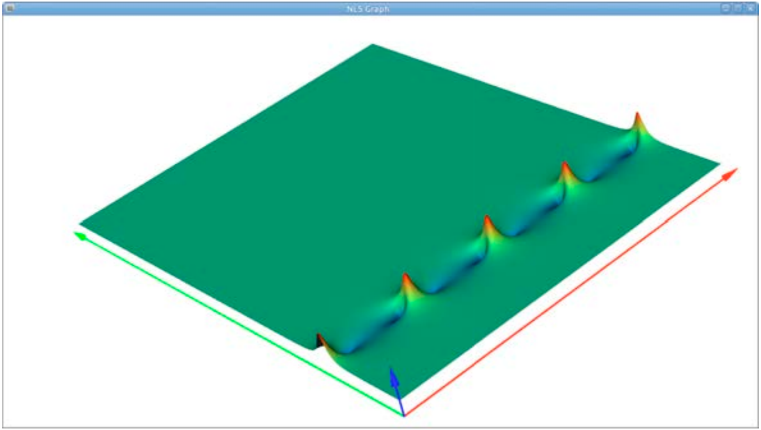
N. N. Akhmediev, V. M. Eleonskii, and N. E. Kulagin, “Exact first order solutions of the Nonlinear Schdinger equation”, *Theor. Math. Phys,* **72**, 809 (1987).

$$\mathcal{A}(x,t;\theta,X,T) =$$
$$= a\, e^{2i|a|^2 t} \cdot \frac{\cosh[\sigma(\theta)(t-T)+2i\theta]+\sin\theta\cos[k(\theta)(x-X)]}{\cosh[\sigma(\theta)(t-T)]-\sin\theta\cos[k(\theta)(x-X)]},$$

$$k_1 = k(\theta) = 2|a|\cos\theta,\ \sigma(\theta) = k(\theta)\sqrt{4|a|^2-k^2(\theta)} = 2|a|^2\sin(2\theta),$$

Akhmediev breathers

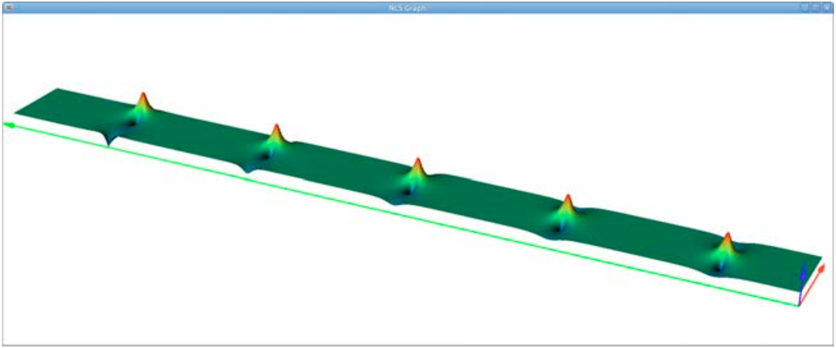
They are spatially periodic and localized in time:



The x coordinate axis marked red, the t coordinate axis marked green. In the future we draw only one period of solution with respect to x.

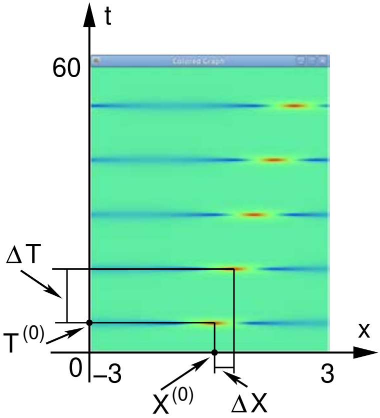
One unstable mode

Generic solution for one unstable mode is well-approximated by a sequence of Akhmediev breathers:



Recurrence of Akhmediev breathers for one unstable mode ($L = 6$).
Here we draw exactly one period in the x-variable.

One unstable mode



Recurrence of Akhmediev breathers for one unstable mode ($L = 6$).

Essential parameters:
First appearance time $T^{(0)}$, position of maximum at first appearance $X^{(0)}$, interval between subsequent appearances ΔT , phase shift between subsequent appearances ΔX.

One unstable mode

Approximation of the genus 2 solution:

$$u(x,t) = \sum_{m=0}^{n} \mathcal{A}\Big(x,t;\phi_1,x^{(m)},t^{(m)}\Big)e^{i\rho^{(m)}} - \frac{1-e^{4in\phi_1}}{1-e^{4i\phi_1}} a e^{2i|a|^2 t}, \quad x \in [0,L],$$

where:

$$x^{(m)} = X^{(1)} + (m-1)\Delta X, \quad t^{(m)} = T^{(1)} + (m-1)\Delta T,$$

$$X^{(1)} = \frac{\arg\alpha}{k_1} + \frac{L}{4}, \quad \Delta X = \frac{\arg(\alpha\beta)}{k_1}, \quad (\bmod\ L),$$

$$T^{(1)} = \frac{1}{\sigma_1}\log\left(\frac{\sigma_1^2}{2|a|^4\epsilon|\alpha|}\right), \quad \Delta T = \frac{1}{\sigma_1}\log\left(\frac{\sigma_1^4}{4|a|^8\epsilon^2|\alpha\beta|}\right),$$

$$\rho^{(m)} = 2\phi_1 + (m-1)4\phi_1, \quad n = \left\lfloor \frac{T-T^{(1)}}{\Delta T} + \frac{1}{2} \right\rfloor,$$

$$\cos\phi_1 = \frac{\pi}{L|a|}, \quad k_1 = \frac{2\pi}{L} = 2|a|\cos(\phi_1), \quad \sigma_1 = k_1\sqrt{4|a|^2 - k_1^2} = 2|a|^2\sin(2\phi_1),$$

$$\alpha = e^{-i\phi_1}\overline{c_1} - e^{i\phi_1}c_{-1}, \quad \beta = e^{i\phi_1}\overline{c_{-1}} - e^{-i\phi_1}c_1.$$

One unstable mode

The spectra curve has genus $g = 2$ and 6 branch points: E_0, E_1, E_2, $\bar{E}_0$, $\bar{E}_1$, $\bar{E}_2$. The pair E_1, E_2 is obtained as a results of splitting the resonant point $\lambda_1 = i|a| \sin \phi_1$:

$$E_l = \lambda_1 + (-1)^l \frac{\epsilon |a|^2}{2\lambda_1} \sqrt{\alpha\beta} + O(\epsilon^2), \quad l = 1, 2,$$

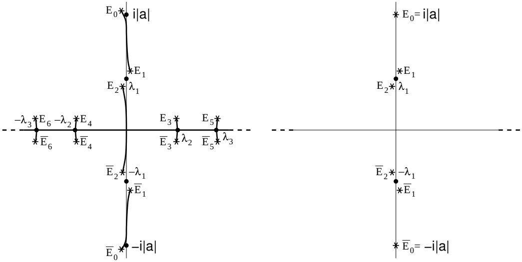


Figure: Right: the exact spectrum; Left: the approximating curve.

Two special symmetric configurations:

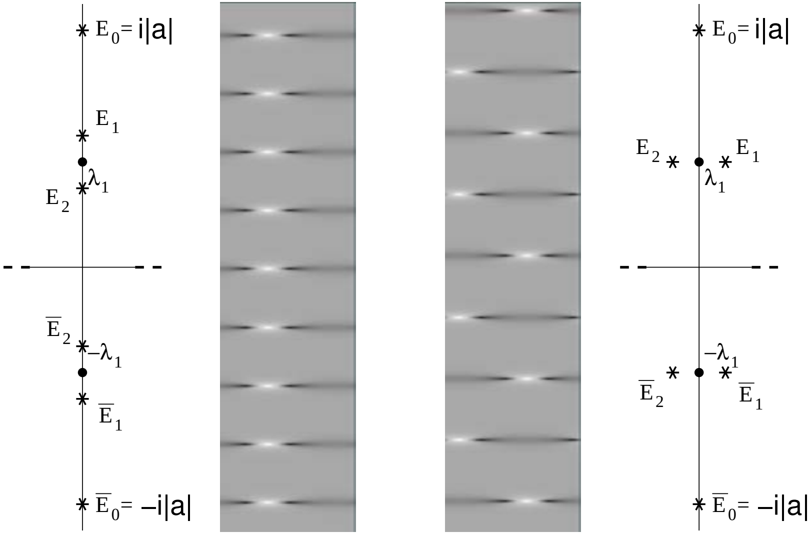


Figure: Left: vertical gap. Right: horisontal gap.

Two special symmetric configurations:

Remark: In these two special cases theta-functions of genus 2 can be reduced to genus 1.

1. Elliptic solutions:

   Akhmediev N.N., Eleonskii V.M, Kulagin N.E., “Exact first-order solutions of the nonlinear Schrödinger equation”, *Theoret. and Math. Phys.*, **72**:2 (1987), 809–818;

2. Reduction for generic parameters:

   Smirnov A.O., “Periodic two-phase “Rogue waves””, *Mathematical Notes*, **94** (2013), 897–907;

3. Reduction in terms of σ-functions:

   Ayano T., Buchstaber V.M., “Relationships between hyperelliptic functions of genus 2 and elliptic functions”, arXiv:2106.06764.

Effect of small loss/gain

As we mentioned above, in real physics it is necessary to take into account small corrections to the NLS equation.
The effect of Hamiltonian perturbations vanishes in the leading order. In contrast, effect of non-Hamiltonian perturbations is non-trivial in the leading order.

Effect of small loss/gain

$$iu_t + u_{xx} + 2u^2\bar{u} = -i\nu u, \quad u = u(x,t), \quad \nu \in \mathbb{R}, \quad |\nu| \ll 1.$$

was recently analytically studied in:

Coppini F., Grinevich P.G., Santini P.M. “The effect of a small loss or gain in the periodic NLS anomalous wave dynamics. I,” *Phys. Rev. E,* **101**:3 (2020), 032204, 8 pages, - Published 6 March 2020; doi:10.1103/PhysRevE.101.032204.

Small loss/gain – experiments and numerics

Our aim was to explain the results of experimental and numerical observations:

O. Kimmoun, H.C. Hsu, H. Branger, M.S. Li, Y.Y. Chen, C. Kharif, M. Onorato, E.J.R. Kelleher, B. Kibler, N. Akhmediev, A. Chabchoub, "Modulation Instability and Phase-Shifted Fermi-Pasta-Ulam Recurrence", *Scientific Reports*, **6**, Article number: 28516 (2016), doi:10.1038/srep28516.

J.M. Soto-Crespo, N. Devine, and N. Akhmediev, "Adiabatic transformation of continuous waves into trains of pulses", *PHYSICAL REVIEW A*, **96**, 023825 (2017).

Small loss/gain – experiments and numerics

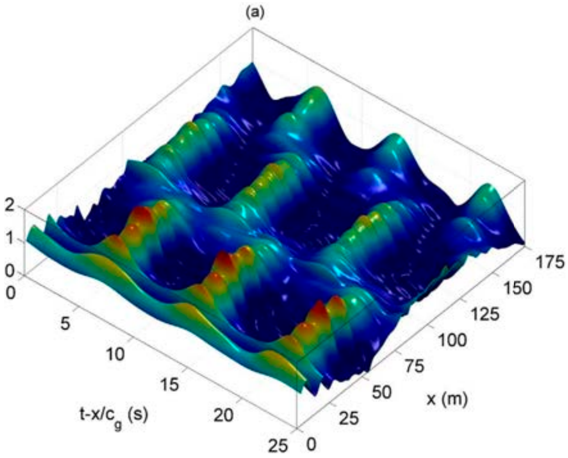


Figure: Measured AB envelope along the large wave facility. The picture was presented in the paper by O. Kimmoun at al, doi:10.1038/srep28516. The phase shift between subsequent appearances of anomalous waves is equal to the semi-period of the wave.

Analytic formulas.

We have the following approximate formulas *the spectral curve is not time–invariant, but it changes each time we have an anomalous wave:*

$$(E_1 - E_2)^2\Big|_{t=0} = -\frac{\epsilon^2|a|^2\alpha\beta}{\sin^2\phi_1},$$

$$(E_1^{(m)} - E_2^{(m)})^2 = -\frac{\epsilon^2|a|^2\alpha\beta}{\sin^2\phi_1} + 4m\nu\cot\phi_1, \quad m \geq 0,$$

where $E_1^{(m)}, E_2^{(m)}$ *are the branch points after the* m^{th}*-th breather.*
Therefore:

$$\Delta X_m := \tilde{x}^{(m+1)} - \tilde{x}^{(m)} = \frac{\arg(Q_m)}{k_1} \pmod{L},$$
$$\Delta T_m := \tilde{t}^{(m+1)} - \tilde{t}^{(m)} = \frac{1}{\sigma_1}\log\left(\frac{\sigma_1^4}{4\epsilon^2|Q_m|}\right),$$

$$\epsilon^2 Q_m = \epsilon^2\alpha\beta - \frac{\nu\sigma_1}{|a|^4}m, \quad m \geq 1, \tag{1}$$

Evolution of the branch points

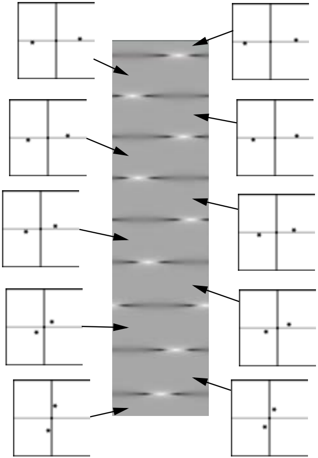
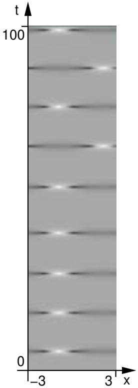


Figure: Evolution of the branch points

Symmetric initial data numerics vs analytics:



$\tilde{t}^{(1)} = 5.51209$ (theory) $\tilde{t}^{(1)} = 5.51208$ (numerics)
$\Delta T_1 = 11.18230$ (theory) $\Delta T_1 = 11.18230$ (numerics)
$\Delta T_2 = 11.40337$ (theory) $\Delta T_2 = 11.40338$ (numerics);
$\Delta T_3 = 11.77375$ (theory) $\Delta T_3 = 11.77376$ (numerics);
$\Delta T_4 = 13.31847$ (theory) $\Delta T_4 = 13.31848$ (numerics);
$\Delta T_5 = 11.84989$ (theory) $\Delta T_5 = 11.84988$ (numerics);
$\Delta T_6 = 11.44140$ (theory) $\Delta T_6 = 11.44142$ (numerics);
$\Delta T_7 = 11.20765$ (theory) $\Delta T_7 = 11.20766$ (numerics);
$\Delta T_8 = 11.04319$ (theory) $\Delta T_8 = 11.04320$ (numerics)

The density plot of $|u(x,t)|$ with $-L/2 \le x \le L/2$, $0 \le t \le 100$, $L = 6$, $\epsilon = 10^{-4}$, $c_1 = 0.3 + 0.4i$, $c_{-1} = \overline{c_1}$, $\nu = 10^{-9}$. After 5 recurrences Q_m changes its sign, from positive to negative values; correspondingly, ΔX_m switches from 0 to $L/2$.

Our analytic formulas as well as numerical simulations provide the strong evidence that:

1. The process of Akhmediev breather generation is very stable;
2. In contrast, the Fermi-Pasta-Ulam-Tsingou recurrence is very sensitive to small perturbation. For example, **small perturbations of solutions generate recurrence.** Moreover, the solutions demonstrate the highest unstability when they reach the maximal value.

At the next slide we illustrate this conclusion by a numeric example.

Stability of Akhmediev breathers – numerical test

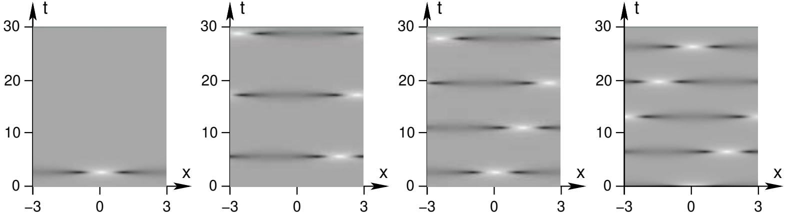


Figure: The figures are enumerated from left to right. We use the recurrence times to measure the effect of perturbation. Smaller recurrence times mean stronger instability. At Figures 2-4 we apply the same perturbation $\delta u(x.0) = 10^{-4}[(0.1 - 0.5i)e^{ik_1 x} + (0.1 + 0.1i)e^{-ik_1 x}]$.

1. Fig.1: Exact Akhmediev breather;
2. Fig.2: Perturbation of the background;
3. Fig.3: Perturbation of Akhmediev breather applied 2.7 seconds before the peak;
4. Fig.4: Perturbation of Akhmediev breather applied at the peak time.

When we started to present these results, we got criticism that our results contradict to the linear stability of N-Akhmediev breather.

There is the following common believe in the literature. Let the NLS background be unstable with respect to the first N modes. Then

1. If $M < N$ unstable modes are exited, then the corresponding M-breather solution is linearly unstable;
2. If all N unstable modes are exited, then the corresponding N-breather solution is neutrally stable, due to “saturation of instabilities”.

Linear perturbation theory near Akhmediev breather

Let us recall the arguments. To study the linear perturbation theory the squared eigenfunctions expansion of the linearized equation is used.

A. Calini, C.M. Schober, "Dynamical criteria for rogue waves in nonlinear Schrödinger models", *Nonlinearity*, **25**:12 (2012) R99–R116; doi:10.1088/0951-7715/25/12/R99.
A. Calini, C.M. Schober, "Observable and reproducible rogue waves", J. Opt. 15 (2013) 105201 (9pp).
A. Calini, C.M. Schober, "Numerical investigation of stability of breather-type solutions of the nonlinear Schrödinger equation", Nat. Hazards Earth Syst. Sci., 14, 14311440, 2014 www.nat-hazards-earth-syst-sci.net/14/1431/2014/doi:10.5194/nhess-14-1431-2014.

Let us recall the main formulas.

To simplify formulas we use the following gauge transformation.

$$u(x,t) \to \exp(2it)u(x,t), \quad \vec{\psi} \to \exp(i\sigma_3 t)\vec{\psi},$$

The new function $u(x,t)$ satisfy:

$$iu_t + u_{xx} + 2|u|^2 u - 2u = 0.$$

Assume for a moment that δu and $\delta\bar{u}$ are independent functions.
Important fact: Squared eigenfunctions

$$\delta u = \psi_1(\lambda,x,t)\varphi_1(\lambda,x,t), \quad \delta\bar{u} = \overline{\psi_2(\lambda,x,t)\varphi_2(\lambda,x,t)}$$

satisfy the complexified linearized NLS equation:

$$\begin{cases} i\,\delta u + \delta u_{xx} + 4u\bar{u}\,\delta u - 2\delta u + 2u^2\,\delta\bar{u} = 0, \\ -i\,\delta\bar{u} + \delta\bar{u}_{xx} + 4u\bar{u}\,\delta\bar{u} - 2\delta\bar{u} + 2\bar{u}^2\,\delta u = 0. \end{cases}$$

(Complexified means exactly that δu and $\delta\bar{u}$ are treated as independent functions.)

Linear perturbation theory near Akhmediev breather

To construct solutions of “normal”, not complexified linearized NLS, it is sufficient to consider “real”linear combination:

$$\left\langle \vec{\psi}(\lambda,x,t),\vec{\varphi}(\lambda,x,t)\right\rangle_{+} := \psi_1(\lambda,x,t)\varphi_1(\lambda,x,t)+\overline{\psi_2(\lambda,x,t)\varphi_2(\lambda,x,t)},$$
$$\left\langle \vec{\psi}(\lambda,x,t),\vec{\varphi}(\lambda,x,t)\right\rangle_{-} := i\left[\psi_1(\lambda,x,t)\varphi_1(\lambda,x,t)-\overline{\psi_2(\lambda,x,t)\varphi_2(\lambda,x,t)}\right]$$

Of course, we have to select spatially-perioic solutions with the period L.

In the aforementioned papers it was shown that if we have N unstable modes and nonlinear superpositions of M Akhmediev breathers, $M \le N$ then

1. If not all unstable modes are exited ($M < N$) then there exist x-periodic squared eigenfunctions exponentially growing in t;
2. If all unstable modes are exited ($M = N$) then all x-periodic squared eigenfunctions are bounded in t;

Therefore the conclusion about linear stability was made.

But we see the intability

Linear perturbation theory near Akhmediev breather

We obtained the following resolution of the paradox (we studied the case $M = N = 1$):

Due to presence of non-removable double points the spectral decomposition of linerized NLS solutions includes not only x-periodic squared eigenfunctions, but also some special combinations of derivatives with respect to the spectral parameter.

The fact that it is necessary to use derivatives with respect to the spectral parameter can be extracted from the paper

I.M. Krichever, Spectral theory of two-dimensional periodic operators and its applications, *Russ. Math. Surv.*, **44**(2), 145225 (1989)

The "missed modes"

The resonant point is:

$$\lambda_1 = \sqrt{\mu_1^2 - 1}, \quad \mu_1 = \frac{k_1}{2} = \frac{\pi}{L},$$

$$k = k_1 = 2\mu_1, \ \ \sigma = \sigma_1 = -4i\lambda_1\mu_1, \ \ \theta(\lambda_1) = \frac{1}{2}\left(ikx - \sigma t\right).$$

Denote

$$\vec{q}(\lambda) = \begin{bmatrix} q_1(\lambda) \\ q_2(\lambda) \end{bmatrix} = \begin{bmatrix} \sqrt{\mu - \lambda}\, e^{\theta(\lambda)} + \sqrt{\mu + \lambda}\, e^{-\theta(\lambda)} \\ \sqrt{\mu + \lambda}\, e^{\theta(\lambda)} - \sqrt{\mu - \lambda}\, e^{-\theta(\lambda)} \end{bmatrix},$$

$$\vec{r}(\lambda) = \begin{bmatrix} r_1(\lambda) \\ r_2(\lambda) \end{bmatrix} = \begin{bmatrix} \sqrt{\mu - \lambda}\, e^{\theta(\lambda)} - \sqrt{\mu + \lambda}\, e^{-\theta(\lambda)} \\ \sqrt{\mu + \lambda}\, e^{\theta(\lambda)} + \sqrt{\mu - \lambda}\, e^{-\theta(\lambda)} \end{bmatrix},$$

$$\vec{\phi}(\lambda) = \begin{bmatrix} \phi_1(\lambda) \\ \phi_2(\lambda) \end{bmatrix} = \begin{bmatrix} 1 \\ (\mu + \lambda) \end{bmatrix} e^{\theta(\lambda)}$$

$$\vec{q} = \vec{q}(\lambda_1), \ \ \vec{r} = \vec{r}(\lambda_1),$$

The Darboux transformation operator is defined by:

$$\mathfrak{D}(\lambda) = (\lambda - \lambda_1)E + \frac{2\lambda_1}{|q_1|^2 + |q_2|^2}\begin{bmatrix} -\overline{q_2} \\ \overline{q_1} \end{bmatrix}[-q_2, q_1],$$

Operator $\mathfrak{D}$ maps the background eigenfunctions to the eigenfunctions for the Akhmediev breather.

$$\vec{\psi}(\lambda) = \mathfrak{D}(\lambda)\vec{\psi}_0(\lambda)$$

Let us consider the following set of dressed eigenfunctions:

$$\vec{\chi}_+(\lambda) = \mathfrak{D}(\lambda)\vec{q}(\lambda), \ \ \vec{\chi}_-(\lambda) = \mathfrak{D}(\lambda)\vec{r}(\lambda), \ \ \tilde{\phi}(\lambda) = \mathfrak{D}(\lambda)\phi(\lambda),$$

Denote:

$$D_\mu = \partial_\mu + \frac{\partial\lambda}{\partial\mu}\partial_\lambda = \partial_\mu + \frac{\mu}{\lambda}\partial_\lambda.$$

We have

$$(D_\mu + D_{\bar{\mu}}) \left\langle \chi_+(\lambda), \chi_-(\lambda) \right\rangle_\pm \Big|_{\lambda=\lambda_1} = \left\langle \chi_+^{(1)}, \chi_-^{(0)} \right\rangle_\pm ,$$

$$(D_\mu + D_{\bar{\mu}})^2 \left\langle \chi_+(\lambda), \chi_-(\lambda) \right\rangle_\pm \Big|_{\lambda=\lambda_1} = \left\langle \chi_+^{(2)}, \chi_-^{(0)} \right\rangle_\pm + 2 \left\langle \chi_+^{(1)}, \chi_-^{(1)} \right\rangle_\pm ,$$

$$(D_\mu + D_{\bar{\mu}}) \left\langle \tilde{\phi}(\lambda), \tilde{\phi}(\lambda) \right\rangle_\pm \Big|_{\lambda=\lambda_1} = 2 \left\langle \tilde{\phi}^{(1)}, \tilde{\phi}^{(0)} \right\rangle_\pm .$$

The "missed modes"

Theorem The following combinations of the derivatives of the squared eigenfunctions with respect to the spectral parameter:

$$Sym_1 = 2l_0^2\left[m_0\left(\left\langle \chi_+^{(2)}, \chi_-^{(0)} \right\rangle_+ + 2\left\langle \chi_+^{(1)}, \chi_-^{(1)} \right\rangle_+\right) - 4\left\langle \chi_+^{(1)}, \chi_-^{(0)} \right\rangle_+ - 16\left\langle \phi_+^{(1)}, \phi_-^{(0)} \right\rangle_+\right],$$

$$Sym_2 = 2l_0^2\left(\left\langle \chi_+^{(2)}, \chi_-^{(0)} \right\rangle_- + 2\left\langle \chi_+^{(1)}, \chi_-^{(1)} \right\rangle_- - \frac{2m_0}{l_0^2}\left\langle \chi_+^{(1)}, \chi_-^{(0)} \right\rangle_-\right),$$

$$\text{where} \qquad \lambda_1 = il_0, \ \ \mu_1 = m_0.$$

- They are x-periodic with period L;
- They exponentially grow as $t \to \pm\infty$;
- They are solutions of the linearized NLS near the Akhmediev breather.

Therefore they represent the "missed modes".

The "missed modes"

Let us shift the solution and the eqgenfunction $x \to x - L/4$. Denote by $\widehat{Sym}_1$ the odd part of Sym_1.

$$\widehat{Sym}_1(x,t) = k\frac{\widehat{Num}_1(x,t)}{\mathcal{D}(x,t)}$$

$$\widehat{Num}_1(x,t) = \Big[\big[48\sigma k^4 - 8\sigma k^6\big]\cosh(\sigma t) + \big[192ik^4 + 8ik^8 - 80ik^6\big]\sinh(\sigma t)\Big]t\sin(kx) +$$

$$+\Big[\big[-16i\sigma + 16i\sigma k^2\big]\cosh(3\sigma t) + \big[-64i\sigma + 40i\sigma k^2 - ik^4\sigma\big]\cosh(\sigma t) +$$

$$+\big[16k^4 - 48k^2\big]\sinh(3\sigma t) + \big[-64k^2 - k^6 + 24k^4\big]\sinh(\sigma t)\Big]\sin(kx) +$$

$$+\Big[\big[-48ik^3 + 64ik + 8ik^5\big]\cosh(2\sigma t) + \big[32\sigma k - 8\sigma k^3\big]\sinh(2\sigma t) +$$

$$+\big[8ik^5 - 48ik^3 + 64ik\big]\Big]\sin(2kx) +$$

$$+\Big[\big[8i\sigma k^2 - i\sigma k^4 - 16i\sigma\big]\cosh(\sigma t) + \big[8k^4 - 16k^2 - k^6\big]\sinh(\sigma t)\Big]\sin(3kx),$$

$$\mathcal{D}(x,t) = 4\left[4k\cosh^2(\sigma t) - 4\sigma\cosh(\sigma t)\cos(kx) + k(4-k^2)\cos^2(kx)\right].$$

Sketch of the proof of completeness

The method was developed in the papers:

For eigenfunctions expansion:
Krichever, I.M., “The spectral theory of “finite-gap” non-stationary Schrödinger operators. The non-stationary Peierls model”, *Functional Anal. Appl.* **20** (1986), 203–214.

For eigenfunctions and squared eigenfunctions expansions:
Krichever, I.M., “Spectral theory of two-dimensional periodic operators and its applications”, *Russian Math. Surveys,* **44**:2 (1989), 145–225.

(The main example was the Kadomtsev-Petviashvili equation.)

For decaying at infinity boundary conditions the completeness of squared eigenfunctions for focusing NLS was proved in:

Kaup, D.J., “Closure of the Squared Zakharov-Shabat Eigenstates”, *Journal of Mathematical Analysis and Applications*, **54** (1976), 849–864.

Sketch of the proof of completeness

How to prove the convergence of the standard Fourier series?

Let us recall how it is done in the calculus textbooks.
Let $u(x) = u(x + 2\pi)$ be a 2π periodic sufficiently regular function.
The partial Fourier series:

$$u_n(x) = \sum_{k=-n}^{n} \hat{u}_j e^{ikx}$$

can be written as:

$$u_n(x) = \int_0^{2\pi} K_n(x-y)u(y)dy, \ \ K_n(x) = \frac{1}{2\pi}\sum_{k=-n}^{n} e^{ikx} = \frac{1}{2\pi}\frac{e^{(n+1/2)ix} - e^{-(n+1/2)ix}}{e^{(1/2)ix} - e^{-(1/2)ix}}$$

The Dirichlet kernel $K_n(x-y)$ admits the following representation:

$$K_n(x-y) = \frac{1}{2\pi}\oint\limits_{|z|=n+1/2} \frac{e^{izx}e^{-izy}dz}{e^{2\pi iz}-1}, \ \ z \in \mathbb{C}.$$

As analogous representation can be obtained in terms of squared NLS eigenfunctions.

Sketch of the proof of completeness

We consider symmetrically normalized eigenfunctions for the Lax pair:

$$\begin{bmatrix}\phi_1(\gamma,x,t)\\ \phi_2(\gamma,x,t)\end{bmatrix} = \frac{1}{\psi_1(\gamma,0,0)+\psi_2(\gamma,0,0)}\begin{bmatrix}\psi_1(\gamma,x,t)\\ \psi_2(\gamma,x,t)\end{bmatrix},$$

which are meromorphic on rational Riemann surface Γ_A with 2 double points. Γ_A is defined by equation

$$\nu^2 = (\lambda^2+|a|^2)(\lambda^2-\lambda_1^2)^2. \quad (2)$$

A point $\gamma\in\Gamma$ is a pair of complex numbers $\gamma=(\lambda,\nu)\in\mathbb{C}^2$, satisfying (2).

We also denote:

$$\nu = (\lambda^2-\lambda_1^2)\,\mu, \quad \text{where } \mu^2=\lambda^2+|a|^2.$$

Let σ and τ be the following involutions on Γ_A

$$\sigma(\lambda,\nu)=(\lambda,-\nu), \quad \tau(\lambda,\nu)=(\bar\lambda,\bar\nu).$$

Spectral curve for the Akhmediev breather

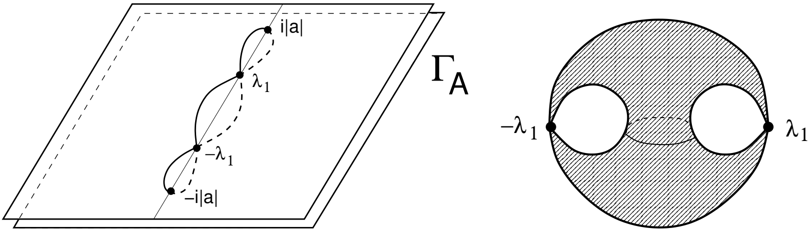


Figure: The curve Γ_A as a two-sheeted covering of the complex plane (left) and its topological model (right).

Γ_A is a two-sheeted covering of the λ-plane:

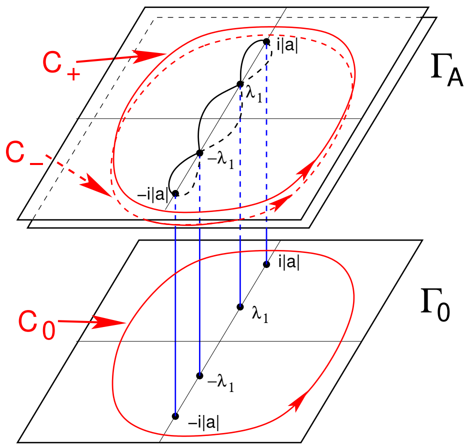
$$\Gamma_A \to \mathbb{C}: \quad (\lambda, \nu) \to \lambda.$$

It has two branch points

$$E_0: \ \lambda = i|a|, \quad \bar{E}_0: \ \lambda = -i|a|,$$

and two double points

Spectral curve for the Akhmediev breather squared eigenfunctions



$\tilde{\Gamma}_A = \Gamma_A \cup \Gamma_0$.
Γ_A is the Akhmediev breather spectral curve. $\Gamma_0 = \mathbb{C}P^1$ is the Riemann sphere.

Blue lines connect the glued pairs of points.
$(\lambda = i, \nu = 0) \leftrightarrow \lambda = i$,
$(\lambda = -i, \nu = 0) \leftrightarrow \lambda = -i$,
$(\lambda = \lambda_1, \nu = 0) \leftrightarrow \lambda = \lambda_1$,
$(\lambda = -\lambda_1, \nu = 0) \leftrightarrow \lambda = -\lambda_1$.

We integrate over $C_+ \cup C_-$. Integrals over C_0 are equal to 0.

Figure: The spectral curve for the squared eigenfunctions $\tilde{\Gamma}_A$ is obtained by gluing Γ_A with the Riemann sphere Γ_0.

The squared eigenfunctions

A vector-function $\vec{\Phi}(\gamma)=\vec{\Phi}(\gamma,x)$ on $\tilde{\Gamma}_A$ is defined by:

$$\vec{\Phi}(\gamma)=\begin{bmatrix}\Phi_1(\gamma)\\ \Phi_2(\gamma)\end{bmatrix}=\begin{cases}\begin{bmatrix}\phi_1^2(\gamma)\\ \phi_2^2(\gamma)\end{bmatrix} & \text{if } \gamma\in\Gamma_A,\\ \\ \begin{bmatrix}\phi_1(\lambda,\nu)\phi_1(\lambda,-\nu)\\ \phi_2(\lambda,\nu)\phi_2(\lambda,-\nu)\end{bmatrix} & \text{if } \gamma\in\Gamma_0.\end{cases}$$

We calculated explicitly the corresponding Cherednik differential we have:

$$\tilde{\Omega}(\gamma)=\begin{cases}\frac{\left[\mu(\lambda^2-\lambda_1^2)+\lambda_1^2+(\mu_1^2+\lambda_1^2)\lambda^2\right]^2}{\mu^2(\lambda^2-\lambda_1^2)^2}d\lambda & \text{if } \gamma\in\Gamma_A,\\ \\ -2\,\frac{\mu^2(\lambda^2-\lambda_1^2)^2+\left[\lambda_1^2+(\mu_1^2+\lambda_1^2)\lambda^2\right]^2}{\mu^2(\lambda^2-\lambda_1^2)^2}d\lambda & \text{if } \gamma\in\Gamma_0.\end{cases}$$

Let us define an analog of the Dirichlet kernel in the Fourier theory.

$$K^{(n)}(x,y) = \frac{1}{\pi} \oint\limits_{C_+\cup C_-} \begin{bmatrix} \Phi_1(\gamma,x) \\ \Phi_2(\gamma,x) \end{bmatrix} \left[-\bar{\Phi}_1(\tau\gamma,y), \quad \bar{\Phi}_2(\tau\gamma,y) \right] \frac{\tilde{\Omega}}{e^{2i\mu L}-1}, \quad (3)$$

$$C_+ \cup C_- = \{(\lambda,\nu) \in \Gamma_A : |\lambda| = R_n\}, \quad R_n = \sqrt{(n\pi/L)^2 - 1} + 1/2.$$

If $n \to \infty$, $K^{(n)}(x,y)$ coincide with the n-th Dirichlet kernel up to $O(1/n)$ corrections. On the other hand, $K^{(n)}(x,y)$ can be calculated as the sum of residues. The integrand in (3) has:

1. First-order poles at the resonant points $\gamma_m = (\lambda_m, \mu_m) =$ $= (\pm\sqrt{(m\pi/L)^2 - 1}, m\pi/L)$, $2 \le m \le n$. The residues at these points are $\Phi_k(\gamma_m, x)\bar{\Phi}_k(\tau\gamma_m, y)$ times normalization constants;
2. Second-order poles at the branch points $\lambda = \pm i, \mu = 0$. The residues at these points are linear combinations of these products and their first derivatives with respect to the spectral parameter;
3. Third-order poles at the double points $\lambda = \pm\lambda_1, \nu = 0$. The residues at these points are linear combinations of these products and their first and second derivatives with respect to the spectral parameter.

Periodicity of the Dirichlet-type kernel

Where we use the properties of the Cherednik differential. We have to check that

$$K^{(n)}(x,y) = \frac{1}{\pi} \oint\limits_{C_+ \cup C_-} \begin{bmatrix} \Phi_1(\gamma,x) \\ \Phi_2(\gamma,x) \end{bmatrix} \left[-\bar{\Phi}_1(\tau\gamma,y), \quad \bar{\Phi}_2(\tau\gamma,y) \right] \frac{\tilde{\Omega}}{e^{2i\mu L} - 1},$$

is L=periodic in x and y. But:

$$K^{(n)}_{lm}(x+L,y) - K^{(n)}_{lm}(x,y) = \frac{(-1)^m}{\pi} \oint\limits_{\gamma \in \Gamma_A, |\lambda| = R_N} \Phi_l(\gamma,x) \bar{\Phi}_m(\tau\gamma,y) \tilde{\Omega},$$