Солитоны и кавитоны в нелокальном уравнении типа Уизема

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Content I

We study solutions of the nonlocal Whitham equation

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} = \frac{\partial}{\partial x}\int\limits_{-\infty}^{\infty} dx' R(x-x')V(x',t) \tag{1}$$

that represents a wide class of equations which are of great interest for nonlinear wave theory. The kernel of the integral term is conventionally defined by the dispersion relation $\omega = k\widetilde{R}(k)$ with

$$R(x) = \int\limits_{-\infty}^{\infty} \widetilde{R}(k)e^{-ikx}dk. \tag{2}$$

Here we examine a particular case of the Whitham equation with a resonance dispersion relation

$$\widetilde{R}(k) = \frac{1}{1-k^2+D^2k^4}. \tag{3}$$

That equation has been proposed to describe nonlinear acoustic waves in simple peristaltic systems (Malkin, 1995). With small D^2 it is also applicable to the waves in a medium with internal oscillators.
We study here specific features of solitary wave solutions to the equation with the dispersion relation presented. It is shown that this equation possesses both smooth and singularity involving solitons with exponential asymptotics, bound states of solitons and solitary waves with oscillating asymptotics. The velocity spectra of exponentially localized solitons turn out to be discrete ones. By solitons, we understand solutions which decay when a spatial variable goes to infinity.

Dimensionless equations are written in a coordinate system moving to the right with the speed of sound in an unperturbed fluid. The independent variables x and t represent phase and time; variable V is proportional to the pressure in the fluid. For peristaltic systems the only parameter D is defined as

$$D^2 = (E_x h^2)/(12(1-v^2)p_S^2 a^2 c_0^4) \tag{4}$$

where E_x, p_s, v are Young modulus, density and Poisson ratio of the shell material, h and a are thickness and radius of the shell, c_0 is the sound speed of the fluid. It should be noted that the form of the nonlocal term in the equation depends on the poles location of the Fourier transform of the kernel. While $D^2 < 1/4$ the poles are located in the real axis so that the causative-type dispersion takes place, $R(x<0)=0$. But for $D^2 > 1/4$, the poles are located symmetrically with respect to $Im(k)=0$, $R(-x)=R(x)$, and dispersion therefore takes a space-type form. The space-type dispersion occurs when the sound velocity of a fluid is greater than the minimum phase velocity of bending oscillations of the shell. This is caused by the origination of radiation physically similar to Cherenkov one.

Hereafter we shall study an ordinary differential equation that is obtained by the inversion of integral operator defined by the equation and transferring to the localized traveling wave solutions.
The result takes the form of the fourth order differential equation

$$D^2 S^{(IV)} + S'' + S = V, \; S = \lambda V + \frac{1}{2} V^2,$$

with traveling coordinate $y = x + \lambda t$, boundary conditions $\lim S(y) = 0$, as $|y| \to \infty$, and parameters D^2, λ.
One feature of this equation is its relation with the class of implicit differential equations. We introduce new variables $u_1 = S, \; u_2 = S'$, $v_1 = -S' - D^2 S'''$, $v_2 = DS''$. Then the equation is reduced to the reversible Hamiltonian system

$$u_1' = u_2, \; v_1' = u_1 - V(u_1), \; Du_2' = v_2, \; Dv_2' = -u_2 - v_1,$$

where $V(u_1)$ is a two-valued function given by solving the equation $2u_1 = 2\lambda V + V^2$ w.r.t. V. The involution is $L : (u_1, v_1, u_2, v_2) \to (u_1, -v_1, -u_2, v_2)$, the fixed point set $Fix(L)$ of L is 2-plane $v_1 = u_2 = 0$.

Since V is two-valued: $V = -\lambda \pm \sqrt{\lambda^2 + 2u_1}$, it is more convenient to consider the space $\mathbb{R}^5$ with coordinates (u_1, v_1, u_2, v_2, V) and its smooth 4-dimensional sub-manifold M given by real solutions of the equation. This is a two-sheeted sub-manifold with respect to the projection $\pi : (u_1, v_1, u_2, v_2, V) \to (u_1, v_1, u_2, v_2)$, its image $\pi(M)$ is half-space $u_1 \geq -\lambda^2/2$. The shape of this sub-manifold is the direct product of a parabola and a 3-plane.

On each sheet (upper one $V(u_1) = -\lambda + \sqrt{\lambda^2 + 2u_1}$ and lower one $V = -\lambda - \sqrt{\lambda^2 + 2u_1}$) the system generates its own differential system. A question arises here: how to conform solutions on two sheets in order to preserve continuity of $S(y)$, when the related orbits hit the boundary of a sheet, i.e. they satisfy the equality $u_1(y_0) = -\lambda^2/2$ and for this y_0 we have $u_2(y_0) \neq 0$?

The rule is: in order to preserve continuity of u_1 we need to use the reversibility of vector fields and make jumps on the boundary of both sheets in accordance to the action of L.
Observe that if an orbit in a sheet does not cross the branching point its behavior is defined by the smooth (in fact – analytic) vector field and any tool working in this case can be used. Our main concern below will be on solutions which are homoclinic orbits to equilibria. These solutions, if they are symmetric ones, can be either smooth or with singularities (they cross the branching plane several times). Smooth homoclinic orbits we call sometimes solitons and those with singularities cavitons.

Smoothing the system. The submanifold M is the graph of the function $u_1 = \lambda V + V^2/2$. Let us rewrite the system in variables (r, v_1, u_2, v_2), $r = V + \lambda$

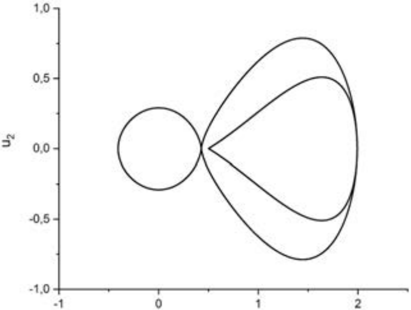
$$rr' = u_2,\ v_1' = \lambda(1 - \lambda/2) - r + r^2/2,\ Du_2' = v_2, Dv_2' = -(u_2 + v_1).$$

The system obtained has singularities along 3-plane $r = 0$ (it is not defined). For upper sheet we get $r = V + \lambda = \sqrt{\lambda^2 + 2u_1} > 0$, but for lower sheet the sign is opposite $r = V + \lambda = -\sqrt{\lambda^2 + 2u_1} < 0$. In order to eliminate the singularity, we multiply equations 2-4 at r and change the "time" to s, $ds = dy/r$, obtaining a smooth differential system

$$\frac{dr}{ds} = u_2,\ \frac{dv_1}{ds} = \lambda(1-\lambda/2)r - r^2 + r^3/2,\ D\frac{du_2}{ds} = rv_2, D\frac{dv_2}{ds} = -r(u_2 + v_1).$$

a) b)

Figure: (a) Graph of a true 2-round soliton-caviton $S(y)$; (b) and its smoothing $r(s)$.



If we put $D=0$ and from 3nd and 4th equations find $u_2=-v_1, v_2=0$, insert this into first two equations we get a slow system.

$$u_1' = -v_1,\ v_1' = u_1 + \lambda \mp \sqrt{\lambda^2 + 2u_1}, \tag{5}$$

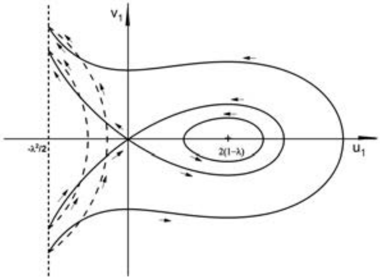


Figure: Slow manifold dynamics with jumps.

Now let us go to the smooth system in variables (r, v_1), here r can take any sign

$$r' = -v_1,\ v_1' = \lambda(1-\lambda/2)r - r^2 + r^3/2.$$

The Hamiltonian of the system is

$$h = \frac{v_1^2}{2} + \frac{\lambda(2-\lambda)r^2}{4} - \frac{r^3}{3} + \frac{r^4}{8}.$$

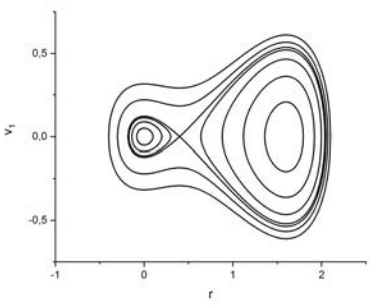


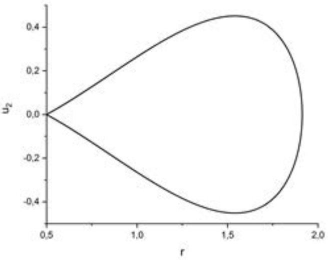
Figure: Phase portrait for smoothen system (**??**).

Types of equilibria for different (D^2, λ).

The system can have different types of equilibria. For us the most interesting are: O (saddle-center) on the upper sheet, it exists when $\lambda \in (0, 1)$;

If D^2 is small (the system is slow-fast one), then slow plane is *almost invariant* elliptic manifold (Gelfreich-Lerman) and finding homoclinic orbits to the saddle-center is a very complicated problem analytically and numerically because of exponentially small splitting of separatrices (look at the slow manifold!).

For $\lambda > 1$ two equilibria exist on the upper sheet, O and another one $P_+ = (2(1-\lambda), 0, 0, 0)$, the latter exists for $1 < \lambda < 2$. The types of these equilibria depend on the value of $D^2 : D^2 > 1/4$, then O is an elliptic point for $1 < \lambda < \lambda_0$, $\lambda_0 = 4D^2/(4D^2 - 1)$, and O is a saddle-focus for $\lambda > \lambda_0$;



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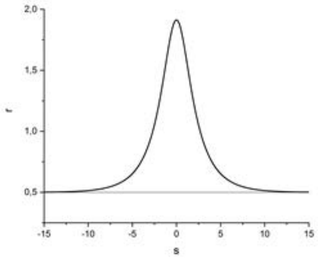
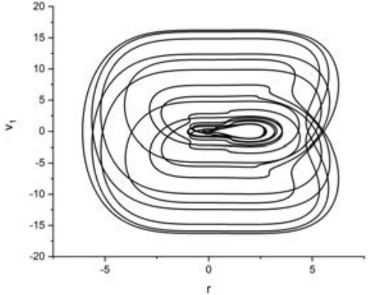


b)

Figure: (a) 1-round homoclinic, $\lambda = 0.5$, $D^2 = 0.23$; (b) Unfolding of this 1-round soliton.

Multi-round solitons (smooth homoclinic orbits) are impossible, but multi-round wave trains are possible.

A situation may occur, when an orbit on an one-dimensional unstable manifold of the saddle-center (which persists under small changes D, λ) gets lie on the stable manifold of some saddle periodic orbit γ in the same level of H. Since the system under consideration is, in addition, reversible, and if saddle-center O and saddle periodic orbit γ are symmetric, then pairing orbit of the stable manifold of the saddle-center gets lie by symmetry on the unstable manifold of γ. Thus, in this case a heteroclinic connection is made up of two heteroclinic orbits, a symmetric saddle-center and a symmetric saddle periodic orbit γ (Lerman-Trifonov).



A very subtle and interesting case arises for small positive $1-\lambda$ near the point $(0,0,0,0)$ on the upper sheet. This equilibrium is degenerate at $\lambda = 1$ with double zero eigenvalue and two imaginary eigenvalues $\pm i\omega$. After scaling the initial equation

$$\lambda = 1-\varepsilon^2,\ \tau = \varepsilon y,\ \frac{d}{dy} = \varepsilon\frac{d}{d\tau},\ S = \varepsilon^2 X,\ D = \frac{\kappa}{\varepsilon}.$$

we come to the following equation that defines the behavior of solutions as $\lambda \to 1-0$

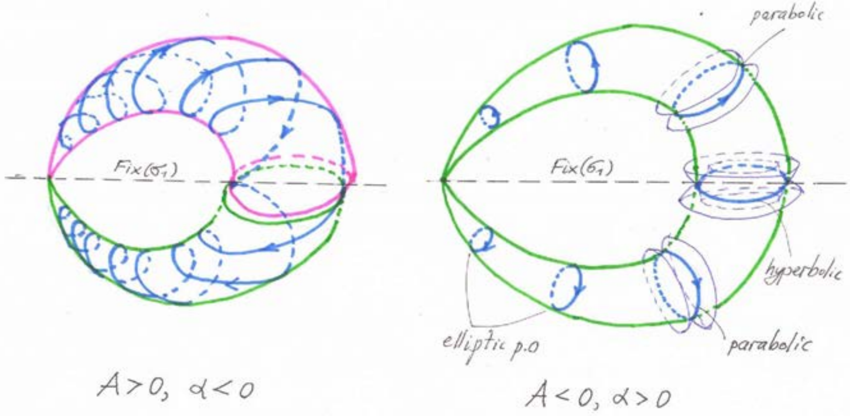
$$\kappa^2\frac{d^4X}{d\tau^4} + \frac{d^2X}{d\tau^2} - X + \frac{1}{2}X^2 - \varepsilon^2 X(1-\frac{3}{2}X+\frac{1}{2}X^2) + \\ \varepsilon^4(-X+3X^2-\frac{5}{2}X^3+\frac{5}{8}X^4)+\cdots.$$

with small parameter κ. Numerical simulations allow us to hypothesize **Hypothesis**. There is a neighborhood of the point $(0,0)$ on the parameter plane (κ,ε) such that a countable set of bifurcation curves exists which correspond to the existence of homoclinic orbits of any roundness.

Homoclinic orbits to saddle-focus

The calculations of equilibria and their types show, in particular, that if $D^2 > 1/4$, then for positive $\lambda > \lambda_0 > 1$ the equilibrium O on the upper sheet is the saddle-focus. The simulations discovered the abundance of symmetric homoclinic orbits to this equilibrium. Is it possible to prove their existence rigorously? We prove the existence of two symmetric homoclinic orbits to the equilibrium O on the upper sheet for small enough $\lambda - \lambda_0$.

Symmetric homoclinic and periodic orbits



To prove this, we need to find the sign of some coefficient A. We find it reducing our problem to that being similar to the problem as for the Swift-Hohenberg equation. To that end, let us scale the traveling coordinate $y = \gamma\xi$, $\gamma = \sqrt{2}D$, in the initial equation of the fourth order. After scaling and dividing at D^2 we get the equation

$$u^{(IV)} + 2u'' + u = (1 - 4D^2 + \frac{4D^2}{\lambda})u - \frac{2D^2}{\lambda^3}u^2 + \frac{2D^2}{\lambda^5}u^3 + \cdots .$$

To calculate the coefficient, we remark that saddle-foci appear as $\lambda > \lambda_0$ as $D^2 > 1/4$. Denote $-\nu = 1 - 4D^2 + \frac{4D^2}{\lambda}$ and consider ν as small positive parameter. After scaling $u = -\kappa u$ with $\kappa = \sqrt{2}D/\lambda^{5/2}$ we come to the equation of the form (we preserve old notations)

$$u^{(IV)} + 2u'' + u = -\nu u + \beta u^2 + u^3 + \cdots . \qquad (6)$$

Let us write the equation in the form of two second order equations

$$u'' + u = v,\ v'' + v = -\nu u + \beta u^2 + u^3 + \cdots .$$

After scaling $u \to \sqrt{\nu}u$, $v \to \nu v$ and denoting $\mu = \sqrt{\nu}$, we get the system

$$u'' + u = \mu v,\ v'' + v = \beta u^2 - \mu(u - u^3) + O(\mu^2).$$

At $\mu = 0$ we have the system whose solutions are of the form

$$u = A\,\mathrm{exp}[i\xi] + \bar{A}\,\mathrm{exp}[-i\xi],\ v = B\,\mathrm{exp}[i\xi] + \bar{B}\,\mathrm{exp}[-i\xi] - \frac{\beta}{3}(A^2\,\mathrm{exp}[2i\xi] + \bar{A}^2\,\mathrm{exp}[-2i\xi]) + 2\beta|A|^2.$$

We add new variables $u' = p, v' = q$, and differentiate these equalities considering them as change of variables $(u, v, p, q) \to (A, \bar{A}, B, \bar{B})$. This gives the expressions for p, q through $A, \bar{A}, B, \bar{B}$. Observe that this change of variables depend 2π-periodically in ξ. Performing this change of variables, we come to the system of four first order differential equations in variables $(A, \bar{A}, B, \bar{B})$ which is the 2π-periodic system in the so-called standard form of the averaging method $X' = \mu F(X, \xi)$ by Krylov-Bogolyubov.

For our case we have

$$A' = -i\mu\frac{B}{2},\ B' = i\mu\frac{A}{2}[1 - \frac{27+2\beta^2}{9}|A|^2],\ \bar{A}' = c.c.,\ \bar{B}' = c.c. \qquad (7)$$

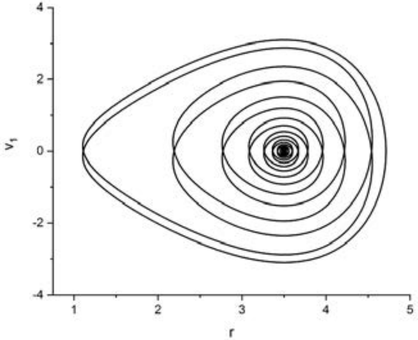
The coefficient we sought for is $\frac{27+2\beta^2}{9}$. It is positive that means the existence of the homoclinic skirt in the system (7) which is integrable and the existence of two symmetric homoclinic orbits in the initial system due to its reversibility (Iooss& Perouemé). The structure of the averaged system is easily restored if introduce real variables (a, b, c, d), $A = a + ib$, $B = c + id$. In these variables we have a Hamiltonian system

$$a' = c,\ c' = \frac{a}{4}(1 - L(a^2+b^2)),\ b' = d,\ d' = \frac{b}{4}(1 - L(a^2+b^2)),\ L = \frac{27+2\beta^2}{9}$$

with Hamiltonian

$$H = \frac{c^2+d^2}{2} - \frac{a^2+b^2}{8} + \frac{L}{16}(a^2+b^2)^2$$

and an additional integral $K = ad - bc$. The common level $H = K = 0$ gives the homoclinic skirt, i.e. one-parameter family of homoclinic orbits to the equilibrium O of a saddle type with merged 2-dimensional stable and unstable manifolds.



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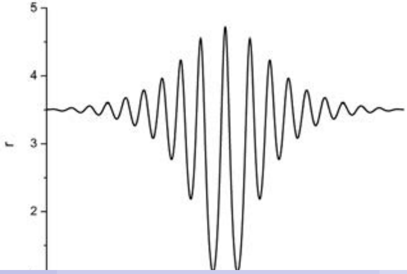

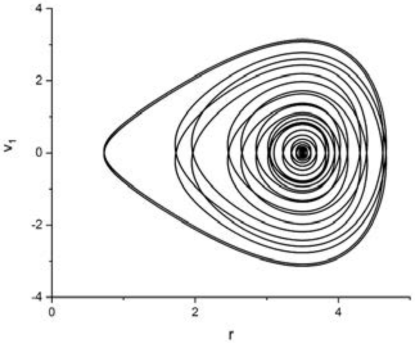

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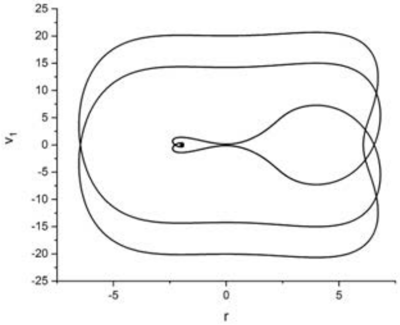

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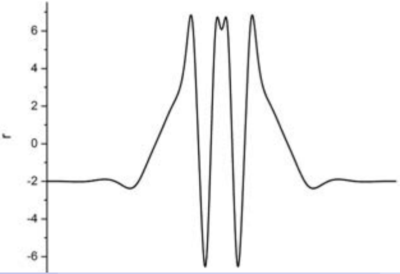

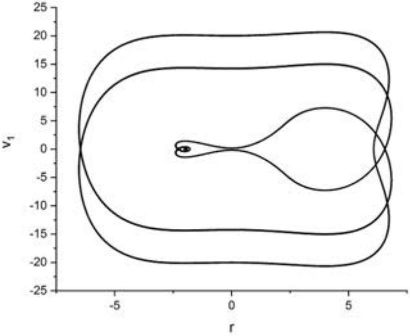

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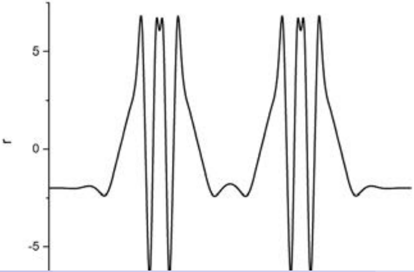

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