Нелинейное уравнение Шредингера и каноническое преобразование

A.I. Dyachenko

Landau Institute for Theoretical Physics
Skolkovo Institute of Science and Technology

Grant RScF 19-72-30028

Summary. Consider 1D NLSE for the field $q(x, t)$:

$$i\dot{q} + \frac{1}{2}q_{xx} + |q|^2 q = 0$$

with periodic boundary condition in space. For Fourier harmonics:

$$i\dot{q}_n - \frac{1}{2}k_n^2 q_n + \sum_{n_1, n_2, n_3} q^*_{n_1} q_{n_2} q_{n_3} \delta_{n+n_1, n_2+n_3} = 0.$$

$$q(x,t) = \sum_n q_n(t) e^{ik_n t}, \quad q_n(t) = \frac{1}{L}\int_0^L q(x,t) e^{-ik_n t} dt, \text{ here } k_n = \frac{2\pi n}{L}.$$

In search of some canonical transformation

$$q_n \to b_n$$

Linear Schrodinger Equation	New Hamiltonian
$i\dot{b}_n - \frac{1}{2}k_n^2 b_n + \lambda_n b_n = 0$	$H = \frac{1}{2}\sum_n k_n^2 \lvert b_n\rvert^2 - \frac{1}{2}\sum_{n,n_1} D_{nn_1} \lvert b_n\rvert^2 \lvert b_{n_1}\rvert^2$

λ_n **does** depend on n and initial conditions and **does not** depend on t

Auxiliary Hamiltonian

For Hamiltonian system with variable $b_n(t)$ the transformation

$$b_n(0) \to b_n(t)$$

is canonical. We will construct this transformation using an auxiliary Hamiltonian $\tilde{H}$(similar to Zakharov, Lvov and Falkovich, 1992)

$$\tilde{H} = \frac{i}{2} \sum_{n,n_1,n_2,n_3} B^{nn_1}_{n_2 n_3} b^*_n b^*_{n_1} b_{n_2} b_{n_3} \delta_{n+n_1,n_2+n_3} +$$
$$+ \frac{i}{3} \sum_{n,n_1,n_2,n_3,n_4,n_5} C^{nn_1 n_2}_{n_3 n_4 n_5} b^*_n b^*_{n_1} b^*_{n_2} b_{n_3} b_{n_4} b_{n_5} \delta_{n+n_1+n_2,n_3+n_4+n_5} + \dots$$

Coefficients in the Hamiltonian have the following properties:

1. Permutations of upper and lower indexes do not change their values
2. $$B^{nn_1}_{n_2 n_3} = -B^{n_2 n_3}_{nn_1} \qquad C^{nn_1 n_2}_{n_3 n_4 n_5} = -C^{n_3 n_4 n_5}_{nn_1 n_2}$$
3. $$B^{nn_1}_{nn_1} = 0 \qquad C^{nn_1 n_2}_{nn_1 n_2} = 0$$

Canonical Transformation

This Hamiltonian helps to calculate coefficients of the series of canonical transformation. One can express old variables q_n in terms of $b_n(0)$:

$$b_n(t) = b_n(0) + t\dot{b}_n(0) + \frac{t^2}{2}\ddot{b}_n(0) + \dots \qquad \dot{b}_n(0) = -i\frac{\delta \tilde{H}}{\delta b_n^*} \qquad \ddot{b}_n(0) = -i\frac{\partial}{\partial t}\frac{\delta \tilde{H}}{\delta b_n^*}$$

$$q_n(0) \quad \Rightarrow \quad b_n(0) \qquad\qquad\qquad t = 1$$

The general form of the canonical transformation in term of power series:

$$\begin{aligned} q_n = b_n &+ \sum_{n_1,n_2,n_3} B^{nn_1}_{n_2n_3} b^*_{n_1} b_{n_2} b_{n_3} \delta_{n+n_1,n_2+n_3} + \\ &+ \frac{1}{2} \sum_{n,n_1,n_2,n_3,n_4,n_5} {}^2B^{nn_1n_2}_{n_3n_4n_5} b^*_{n_1} b^*_{n_2} b_{n_3} b_{n_4} b_{n_5} \delta_{n+n_1+n_2,n_3+n_4+n_5} + \\ &+ \sum_{n,n_1,n_2,n_3,n_4,n_5} C^{nn_1n_2}_{n_3n_4n_5} b^*_{n_1} b^*_{n_2} b_{n_3} b_{n_4} b_{n_5} \delta_{n+n_1+n_2,n_3+n_4+n_5} + \dots \end{aligned}$$

We seek for the best choice for $B^{nn_1}_{n_2n_3}$, $C^{nn_1n_2}_{n_3n_4n_5}$, etc.

Resonant Manifolds

The Hamiltonian takes (after symmetrization) the following form:

$$
\begin{aligned}
H &= \frac{1}{2}\sum_n k_n^2 |b_n|^2 - \frac{1}{2}\sum_{n,n_1,n_2,n_3} T^{nn_1}_{n_2 n_3} b_n^* b_{n_1}^* b_{n_2} b_{n_3} \underline{\delta_{n+n_1,n_2+n_3}} - \\
&- \frac{1}{6}\sum_{n,n_1,n_2,n_3,n_4,n_5} T^{nn_1 n_2}_{n_3 n_4 n_5} b_n^* b_{n_1}^* b_{n_2}^* b_{n_3} b_{n_4} b_{n_5} \underline{\delta_{n+n_1+n_2,n_3+n_4+n_5}} + \dots
\end{aligned}
$$

Here

$$
\begin{aligned}
T^{nn_1}_{n_2 n_3} &= 1 - \frac{1}{2}(k_n^2 + k_{n_1}^2 - k_{n_2}^2 - k_{n_3}^2) B^{nn_1}_{n_2 n_3} \\
T^{nn_1 n_2}_{n_3 n_4 n_5} &= R^{nn_1 n_2}_{n_3 n_4 n_5} - (k_n^2 + k_{n_1}^2 + k_{n_2}^2 - k_{n_3}^2 - k_{n_4}^2 - k_{n_5}^2) C^{nn_1 n_2}_{n_3 n_4 n_5}
\end{aligned}
$$

$$
R_4 \Leftarrow \begin{cases} k_n^2 + k_{n_1}^2 - k_{n_2}^2 - k_{n_3}^2 = 0 \\ n + n_1 - n_2 - n_3 = 0. \end{cases}
$$

$$
R_6 \Leftarrow \begin{cases} k_n^2 + k_{n_1}^2 + k_{n_2}^2 - k_{n_3}^2 - k_{n_4}^2 - k_{n_5}^2 = 0 \\ n + n_1 + n_2 - n_3 - n_4 - n_5 = 0. \end{cases}
$$

$T^{nn_1}_{n_2n_3} = 1 - \frac{1}{2}(k_n^2 + k_{n_1}^2 - k_{n_2}^2 - k_{n_3}^2)B^{nn_1}_{n_2n_3}$

$$\sum_{n,n_1,n_2,n_3} T^{nn_1}_{n_2n_3} b_n^* b_{n_1}^* b_{n_2} b_{n_3} = \sum_{n_i \in R^4} T^{nn_1}_{n_2n_3} b_n^* b_{n_1}^* b_{n_2} b_{n_3} + \cancel{\sum_{n_i \notin R^4} T^{nn_1}_{n_2n_3} b_n^* b_{n_1}^* b_{n_2} b_{n_3}}$$

The best choice for $B^{nn_1}_{n_2n_3}$ is:

$$\begin{aligned} B^{nn_1}_{n_2n_3} &= \frac{2\cdot 1}{k_n^2 + k_{n_1}^2 - k_{n_2}^2 - k_{n_3}^2} && \text{if } n_i \notin R^4, \\ B^{nn_1}_{n_2n_3} &= 0 && \text{if } n_i \in R^4 \end{aligned}$$

R^4 has only trivial solutions:

$$n = n_2, \quad n_1 = n_3 \qquad\qquad n = n_3, \quad n_1 = n_2$$

$$T^{nn_1}_{n_2n_3} \quad \Rightarrow D_{nn_1} \quad = \begin{cases} 2 & \text{if } n \neq n_1 \leq 0, \\ 1 & \text{if } n = n_1. \end{cases}$$

Note: $B^{nn_1}_{n_2n_3}$ satisfies symmetry conditions and has no singularities.

$T^{nn_1n_2}_{n_3n_4n_5} = R^{nn_1n_2}_{n_3n_4n_5} - (k_n^2 + k_{n_1}^2 + k_{n_2}^2 - k_{n_3}^2 - k_{n_4}^2 - k_{n_5}^2) C^{nn_1n_2}_{n_3n_4n_5}$

$$\begin{aligned}\sum_{n_i} T^{nn_1n_2}_{n_3n_4n_5} b_n^* b_{n_1}^* b_{n_2}^* b_{n_3} b_{n_4} b_{n_5} &= \sum_{n_i \in R^6} T^{nn_1n_2}_{n_3n_4n_5} b_n^* b_{n_1}^* b_{n_2}^* b_{n_3} b_{n_4} b_{n_5} + \\ &+ \cancel{\sum_{n_i \notin R^6} T^{nn_1n_2}_{n_3n_4n_5} b_n^* b_{n_1}^* b_{n_2}^* b_{n_3} b_{n_4} b_{n_5}}\end{aligned}$$

The best choice for $C^{nn_1n_2}_{n_3n_4n_5}$ is:

$$\begin{aligned} C^{nn_1n_2}_{n_3n_4n_5} &= \frac{R^{nn_1n_2}_{n_3n_4n_5}}{k_n^2 + k_{n_1}^2 + k_{n_2}^2 - k_{n_3}^2 - k_{n_4}^2 - k_{n_5}^2} \qquad &\text{if } n_i \notin R^6, \\ C^{nn_1n_2}_{n_3n_4n_5} &= 0 &\text{if } n_i \in R^6 \end{aligned}$$

R^6 **has** nontrivial solutions.

$$T^{nn_1n_2}_{n_3n_4n_5} = \begin{cases} R^{nn_1n_2}_{n_3n_4n_5} & \text{if } n_i \in R^6, \\ 0 & \text{if } n_i \notin R^6. \end{cases}$$

Note: $C^{nn_1n_2}_{n_3n_4n_5}$ satisfies symmetry conditions and has no singularities.

New Hamiltonian

Six waves interactions

$$R^{nn_1n_2}_{n_3n_4n_5} \equiv 0 \qquad \text{if } n_i \in R^6$$

$$H = \frac{1}{2}\sum_n k_n^2|b_n|^2 - \frac{1}{2}\sum_{n,n_1} D_{nn_1}|b_n|^2|b_{n_1}|^2 - \cancel{\frac{1}{6}\sum_{n_i\in R^6} R^{nn_1n_2}_{n_3n_4n_5}\, b_n^* b_{n_1}^* b_{n_2}^* b_{n_3} b_{n_4} b_{n_5}} + \ldots$$

and the equation for b:

$$i\dot{b}_n - \frac{1}{2}k_n^2 b_n + \Big[\sum_{n_1} D_{nn_1}|b_{n_1}|^2\Big] b_n = 0$$

Here

$$\sum_{n_1} D_{nn_1}|b_{n_1}|^2 = \lambda_n$$

does not depend on t and is defined by initial conditions only.

The solution of this equation is simple (action-angle):

$$b_n(t) = b_n(0)e^{i(\lambda_n - \frac{1}{2}k_n^2)t}$$

Weakly Nonlinear Case $\frac{b^2}{k^2} << 1$

$$i\dot{q} + \frac{1}{2}q_{xx} + |q|^2 q = 0$$

For Fourier harmonics:

$$i\dot{q}_n - \frac{1}{2}k_n^2 q_n + \sum_{n_1,n_2,n_3} q_{n_1}^* q_{n_2} q_{n_3} \delta_{n+n_1,n_2+n_3} = 0.$$

Canonical Transformation

$$q_n = b_n + \sum_{k_n^2+k_{n_1}^2 \neq k_{n_2}^2+k_{n_3}^2} \frac{b_{n_1}^* b_{n_2} b_{n_3}}{(k_{n_1}-k_{n_2})(k_{n_1}-k_{n_3})}\delta_{n+n_1,n_2+n_3} \qquad \frac{b^2}{k^2} << 1$$

$$H = \frac{1}{2}\sum_n k_n^2 |b_n|^2 - \frac{1}{2}\sum_{n,n_1} D_{nn_1}|b_n|^2|b_{n_1}|^2$$

$$i\dot{b}_n - \frac{1}{2}k_n^2 b_n + \lambda_n b_n = 0 \qquad\qquad b_n(t) = b_n(0)e^{i(\lambda_n - \frac{1}{2}k_n^2)t}$$

$\lambda_n = \left[\sum_{n_1} D_{nn_1}|b_{n_1}|^2\right]$ = constant - is almost Number of Waves

Numerics (forward) $L_d/L_{nl} = 0.05$

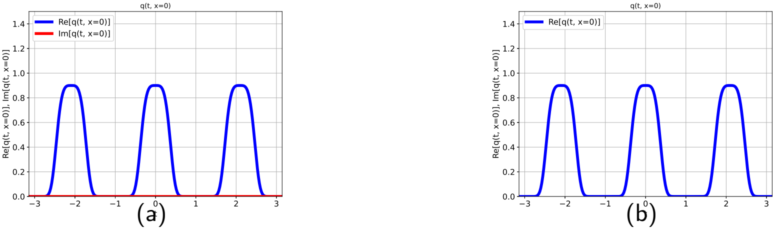
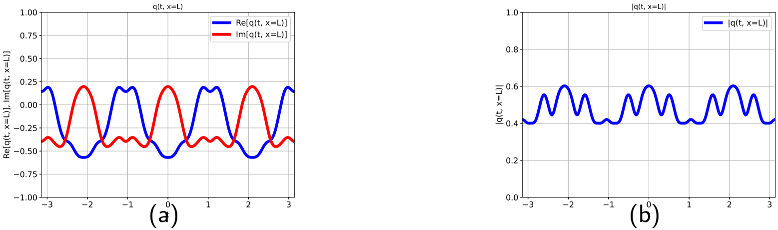


Figure: (a) – Re and Im of the signal, (b) – $|q|$



Numerics (backward)

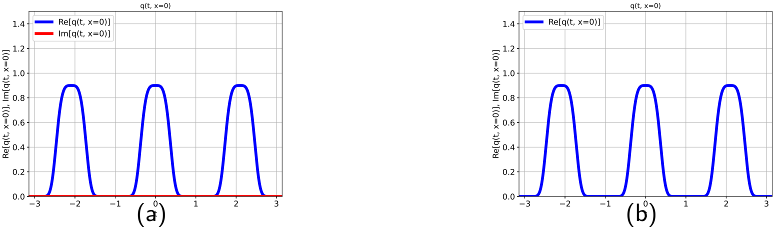
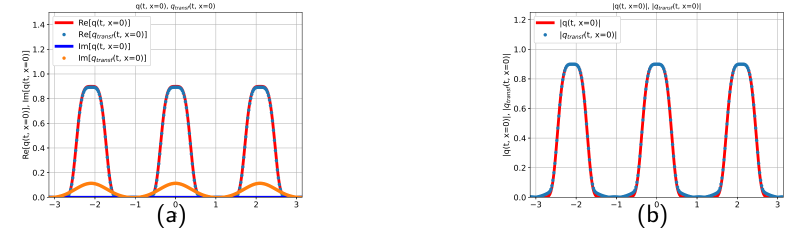
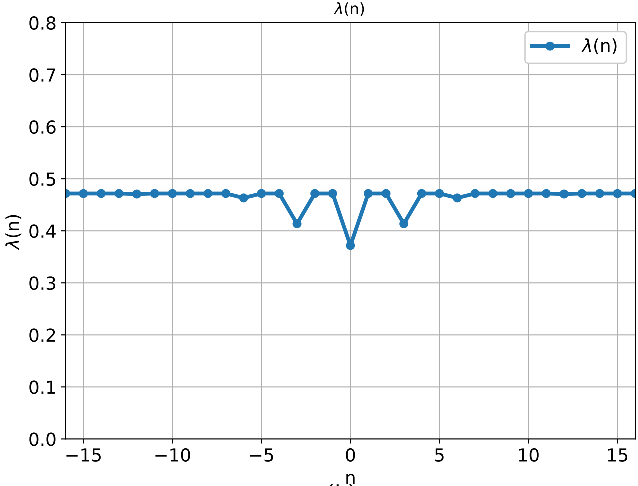


Figure: (a) – Re and Im of the signal, (b) – $|q|$



λ_n



(b)

Figure: λ_n