Статистические свойства множественных столкновений солитонов

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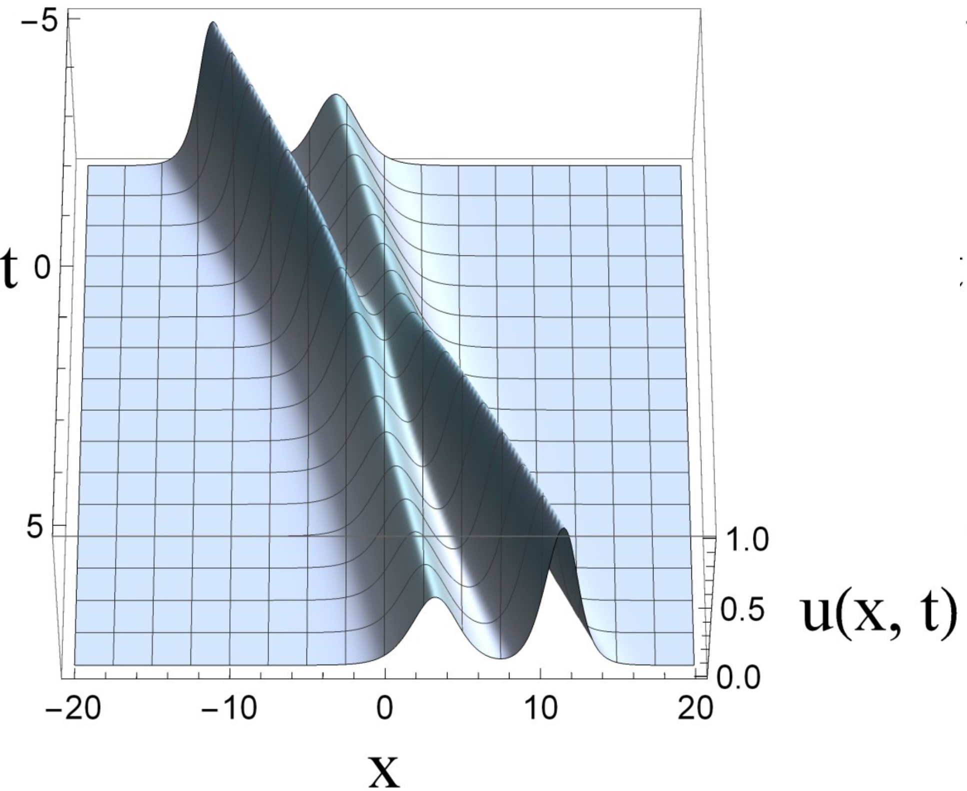
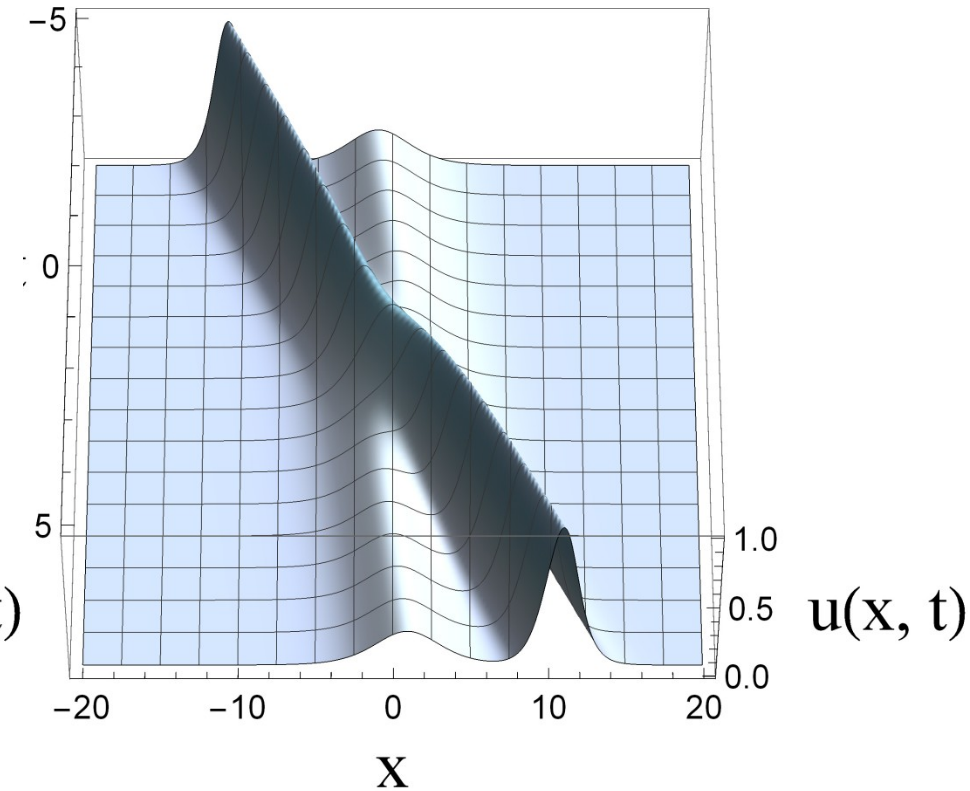
Soliton interactions

Solition interactions represent a complicated nonlinear wave phenomenon which may be observed in various physical applications.
The process may be described exactly using analytical solutions within the integrable (approximate) equations.

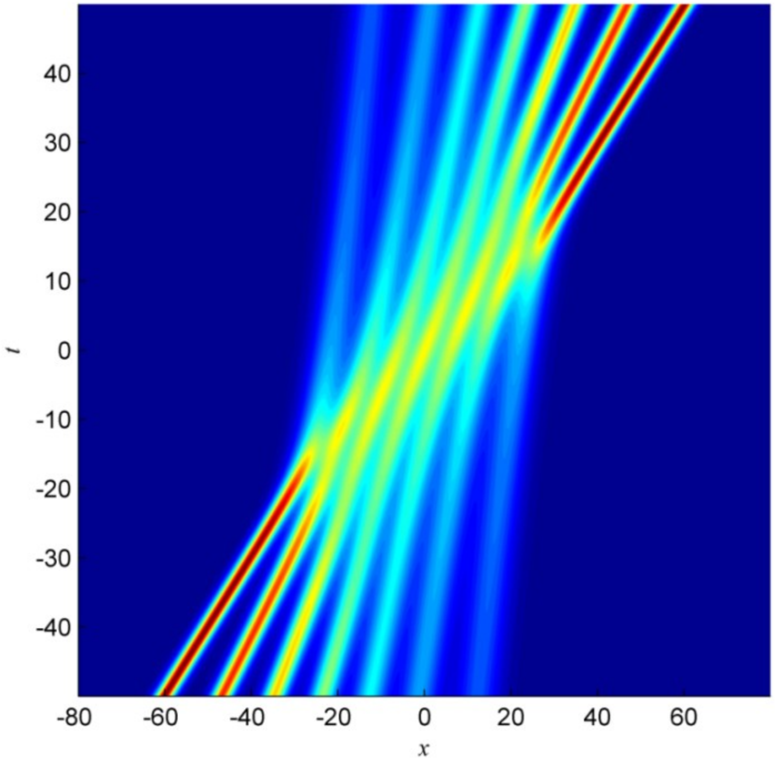
xchange interaction of KdV solitons



Overtaking interaction of KdV solitons

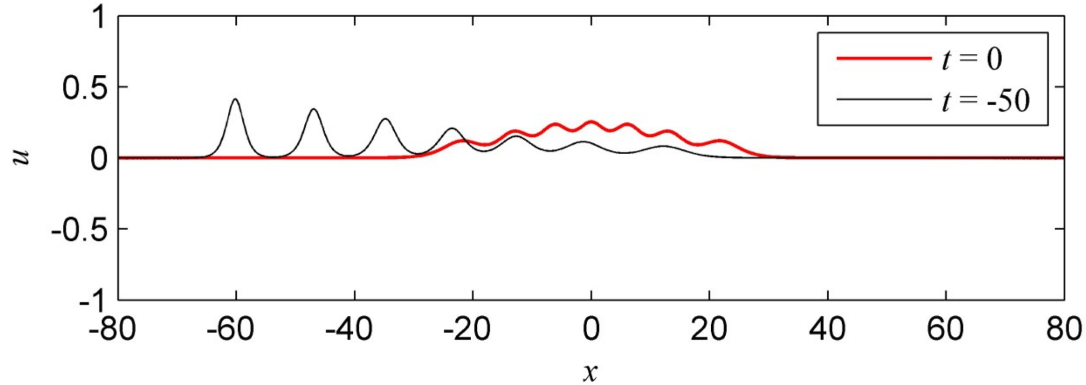


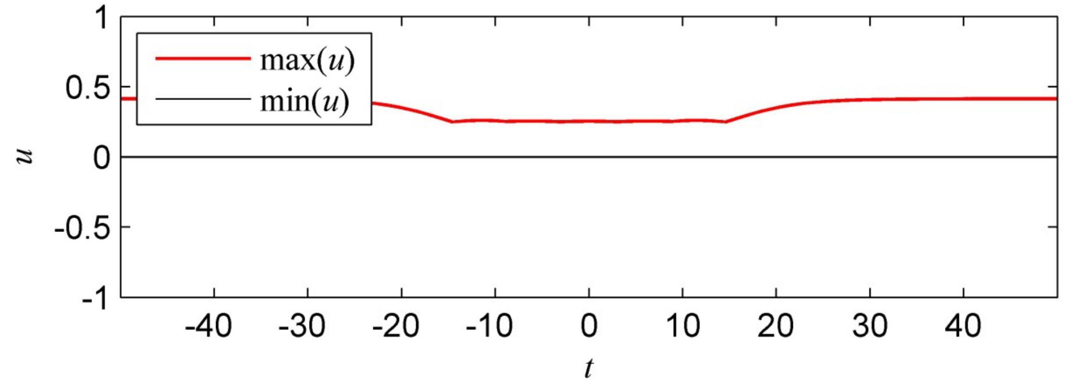
Soliton interactions



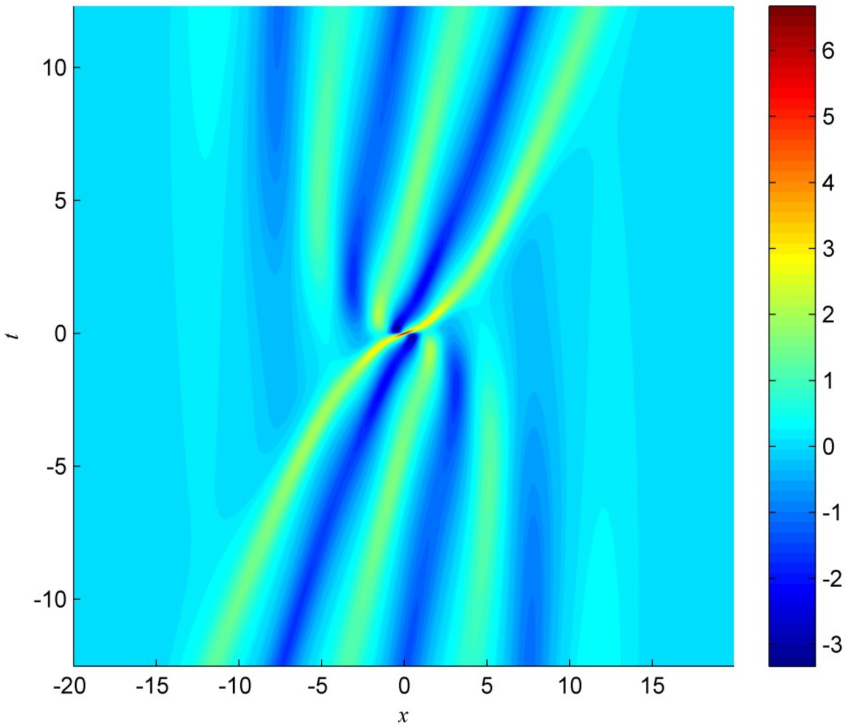
Interaction of 7 KdV solitons *u*(*x*,*t*)

Surprisingly, collisions of may KdV solitons lead to the decrease of wave amplitudes
in the focusing area.
This situation is general for solitions of the same sign (mKdV, GE).



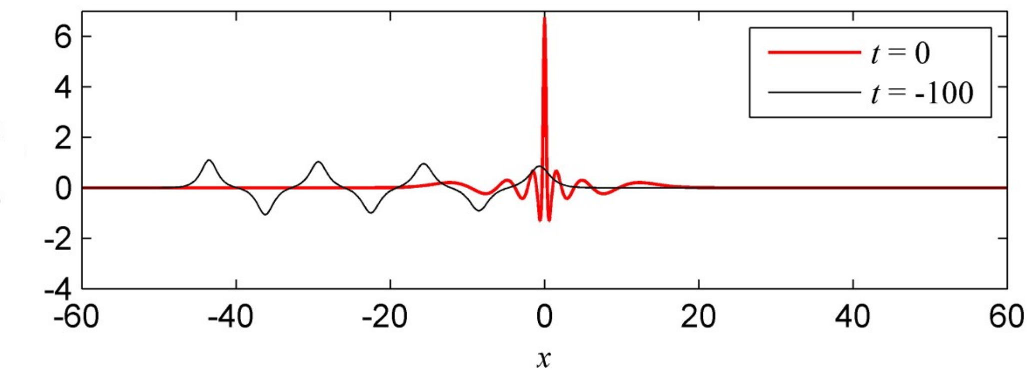


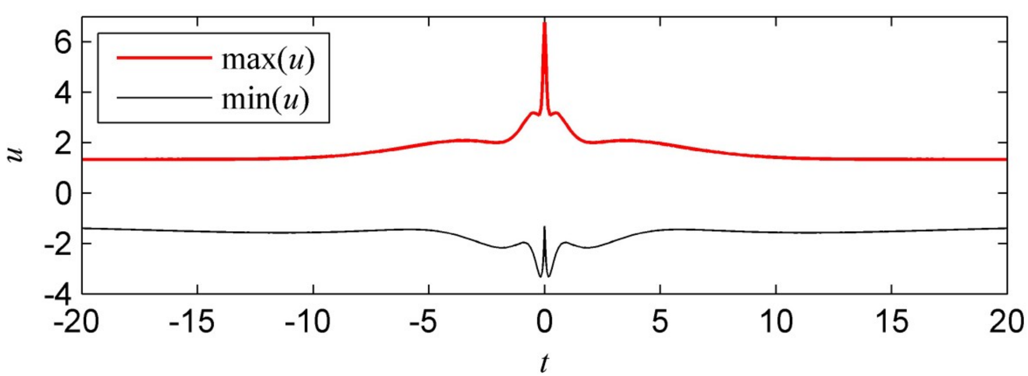
Soliton interactions



Interaction of 7 mKdV solitons
of alternating signs

However, the constructive nonlinear interference between solitons occur when their signs are alternating (mKdV, GE)
or phases are π-shifted (NLSE).





[Slunyaev & Pelinovsky, 2016; Slunyaev, 2019]

Soliton interactions

The standard definition for the variance reads $\sigma^2 = \overline{(u - \bar{u})^2} = \overline{u^2} - \bar{u}^2 > 0$

Where the overline has the meaning of averaging over the spatial interval.

For *N* KdV solitons with amplitudes $A_j = 2k_j^2$, $j = 1, \ldots, N$, confined within the interval *L* the integrals may be estimated analytically, when the solitons do not overlap:

$$\bar{u} = \frac{1}{L}\int_L u\,dx \approx \frac{1}{L}\sum_{j=1}^{N} 4k_j = 4\frac{N}{L}\langle k\rangle = 4\rho\langle k\rangle$$

$$\overline{u^2} = \frac{1}{L}\int_L u^2 dx \approx \frac{1}{L}\sum_{j=1}^{N}\frac{16}{3}k_j^3 = \frac{16}{3}\frac{N}{L}\langle k^3\rangle = \frac{16}{3}\rho\langle k^3\rangle$$

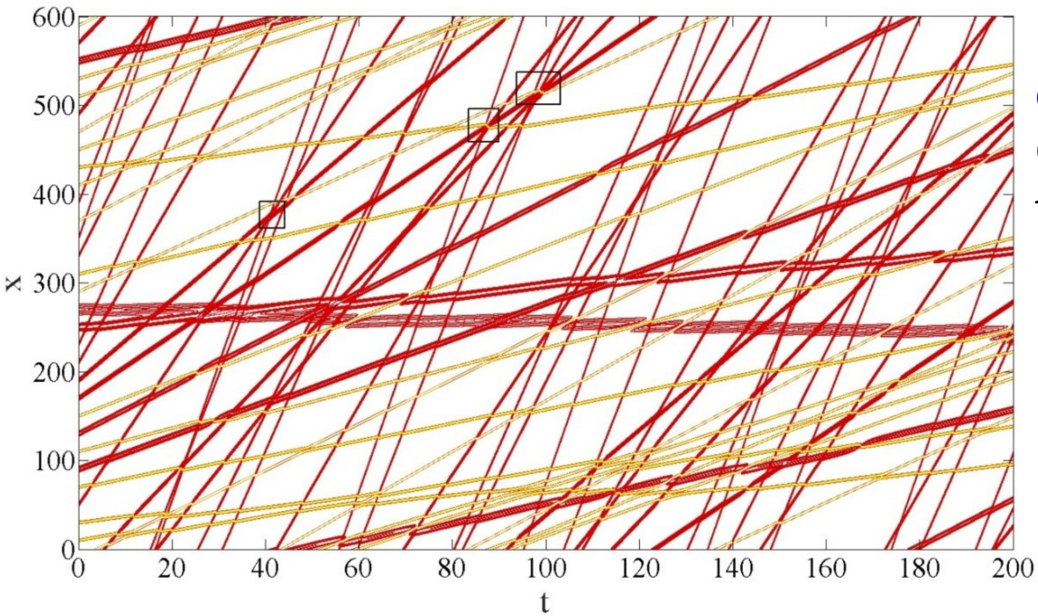
Here the angle brackets denote averaging over the soliton parameters (in the spectral plane), and the quantity x = *N*/*L* has the meaning of the solition density.

The request of non-negative variance ?² leads to the condition on a maximal soliton density: $\rho \le \rho_{cr}, \quad \rho_{cr} = \frac{\langle k^3\rangle}{3\langle k\rangle^2}$

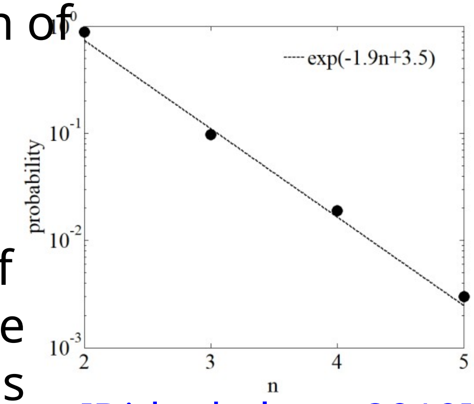
Thus, the case of many densely located solitons is in some sense extreme even when the solitons have similar signs (and then the relation between the mean of *u* and the density x holds). This should be relevant to the description of a 'soliton condensate' state.

[Pelinovsky & Shurgalina, 2016; El, 2016]

Soliton interactions



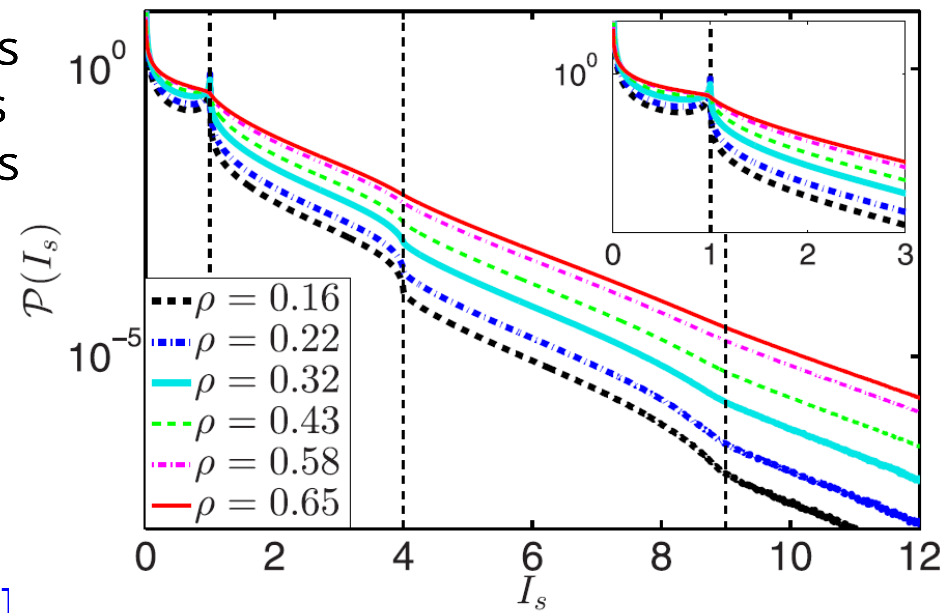
Boxes show multiple collisions within the of Gardner equation of focusing type



Probability of multiple collisions

[Didenkulova, 2019]

Gas of NLS envelope solitons with equal amplitudes but different speeds



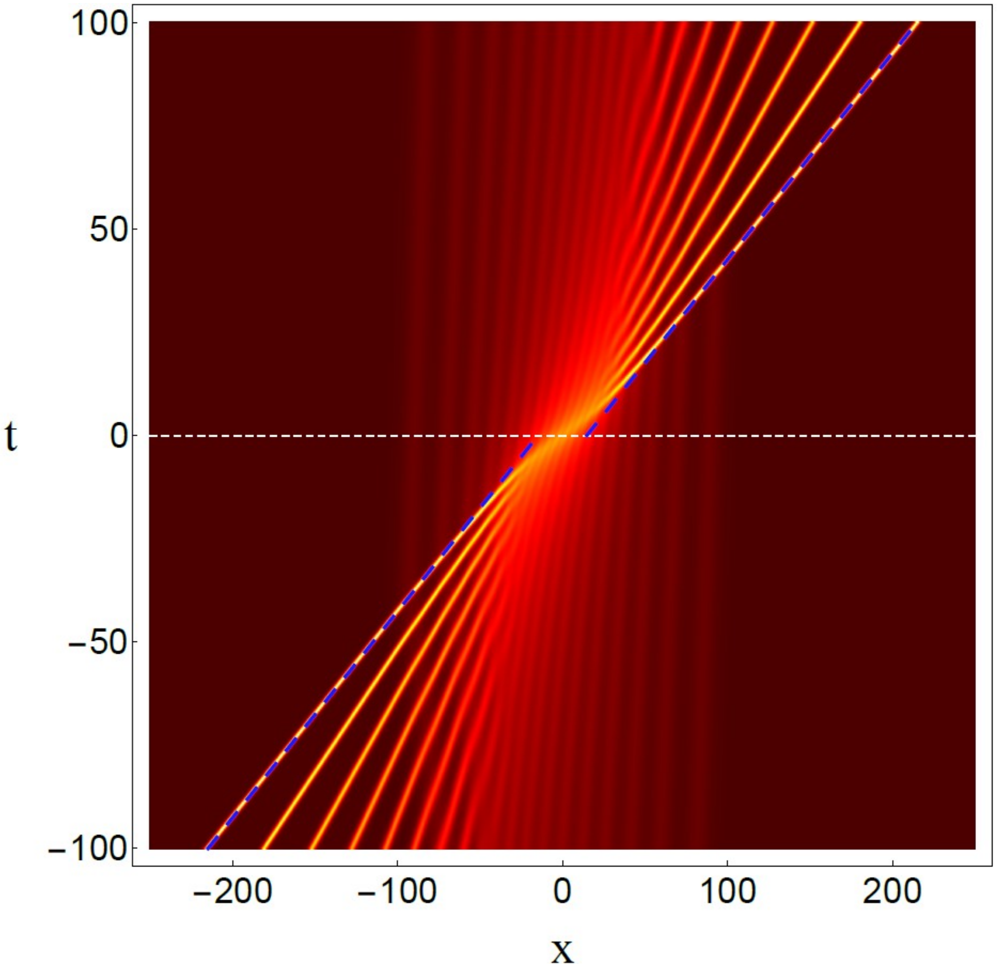
Multiple collisions contribute to the probability distribution function

ielash & Agafontsev, 2018; Akhmediev et al, 2009]

Problem setup

We consider Interactions of KdV solitons with amplitudes decaying according to the geometric progression: $A_j = 1/d^{(j-1)}$, $j = 1, 2, \ldots, N$, where N is large. This set corresponds to the distribution of eigenvalues of the scattering problem (represented by the stationary Schrödinger equation) for a parabolic potential, which may serve as a first approximation to any

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$



The N-soliton solution is constructed in Wolfram Matematica using the Darboux transformation and 100-digits arithmetic

$$u_N(x,t) = -2\frac{\partial^2}{\partial x^2}\ln W_N(\psi_1,\psi_2,\ldots,\psi_N)$$

$$A_j = \frac{1}{d^j}, \quad j = 1,\ldots,N$$

Synchronous collisions at $x = 0$ and $t = 0$ are concerned which possess the symmetry $u_N(-x,-t) = u_N(x,t)$.

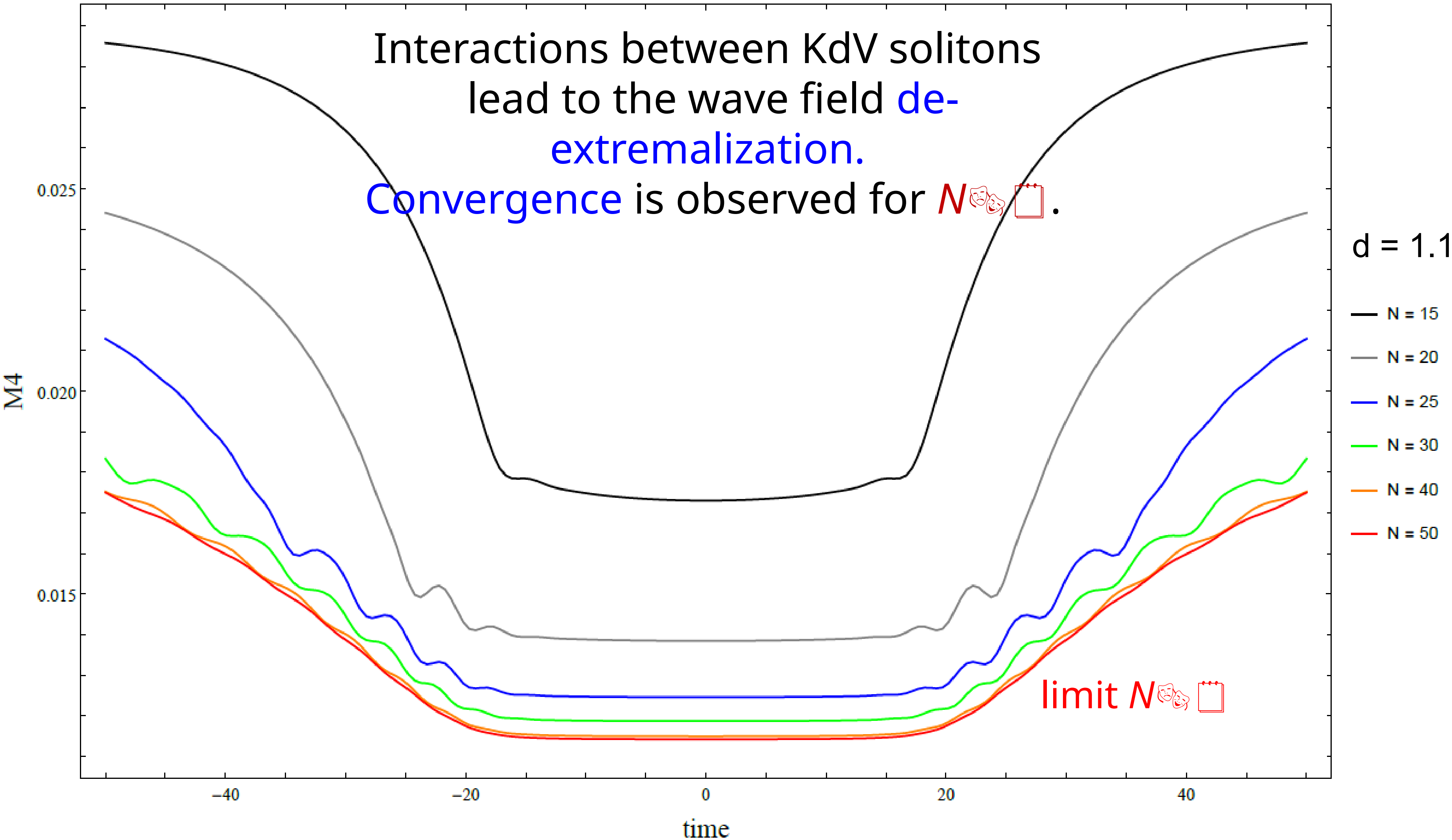
Evolution of statistical moments

A quasi-stationary state is observed when looking at the third and fourth statistical moments,

$$\mu_n = \int_{-\infty}^{\infty} u^n dx,$$

Interactions between KdV solitons lead to the wave field de-extremalization.

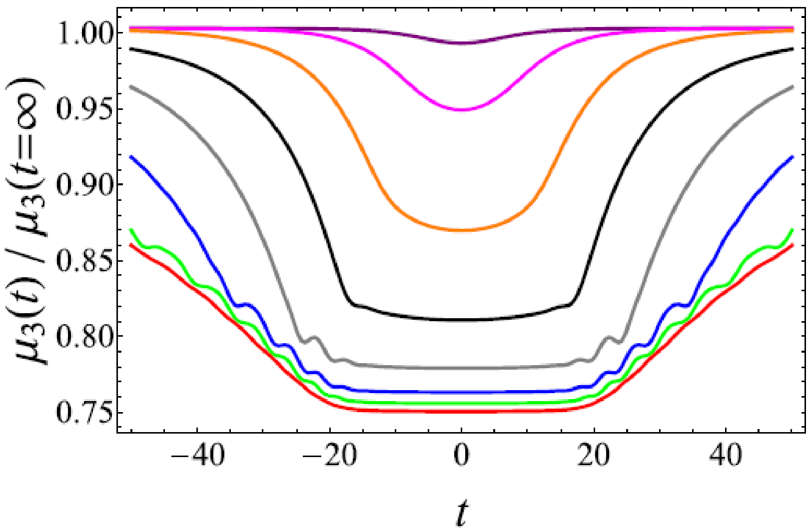
Convergence is observed for $N \to \infty$.



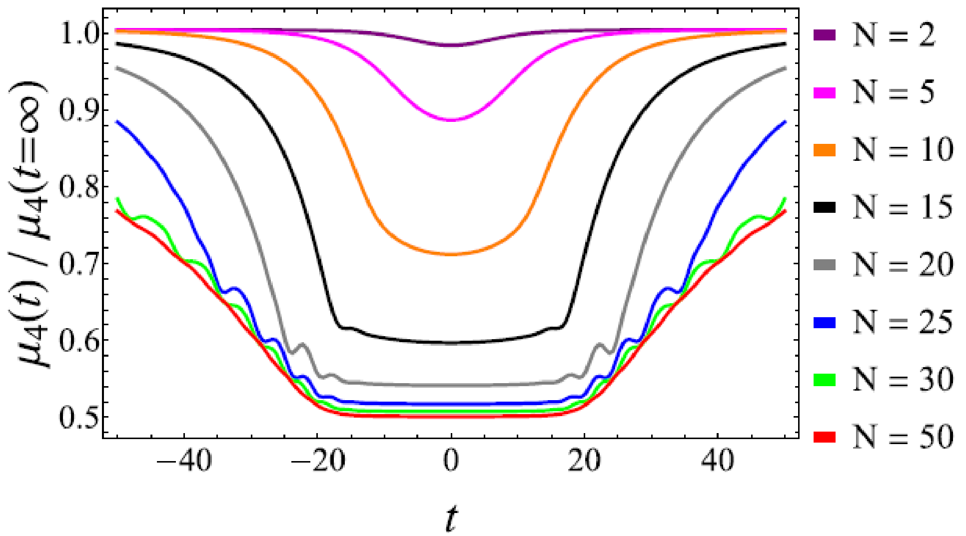
Evolution of statistical moments

d = 1.1

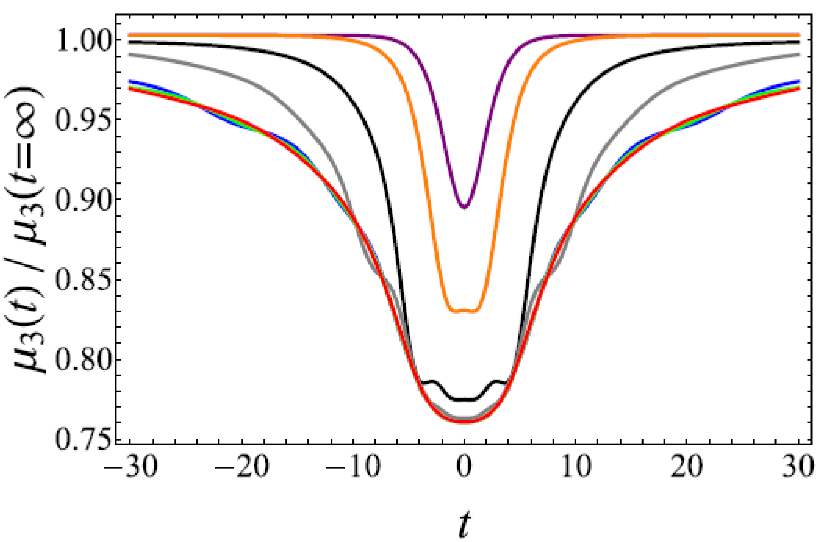
asymmetry

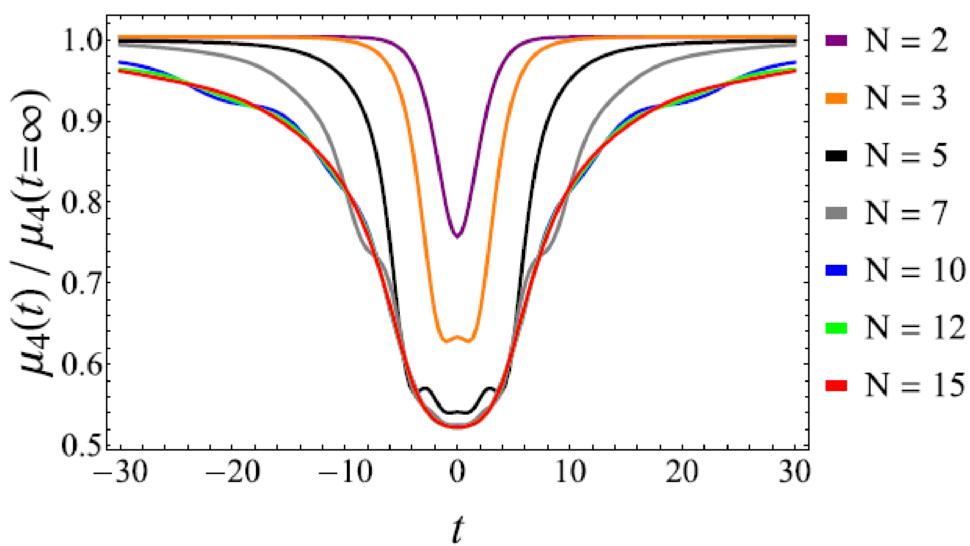


kurtosis



d = 1.6





Reduction of statistical moments

The KdV equation possesses an infinite set of conserved quantities:

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \qquad \frac{d}{dt} I_n = 0, \quad I_n = \int_{-\infty}^{\infty} T_n(u)\,dx, \quad n = 1,2,\ldots$$

The densities in the form of Miura et al (1968) will be used hereafter:

$$I_1 = 6\int_{-\infty}^{\infty} u\,dx \qquad I_2 = 18\int_{-\infty}^{\infty} u^2\,dx \qquad I_3 = 72\int_{-\infty}^{\infty}\left(u^3 - \frac{1}{2}(u_x)^2\right)dx \qquad I_4 = \ldots$$

The quantities may be written in form (Karpman, 1975) $I_n = \frac{6^n}{m}\mu_n\left(1 + O(\varepsilon^{-1})\right)$

where $\mu_n = \int_{-\infty}^{\infty} u^n\,dx$

when the similarity parameter (Ursell number) ε is assumed large, ε >> 1. This parameter may be estimated, for example, via the balance between the two contributors to the integral I_3:

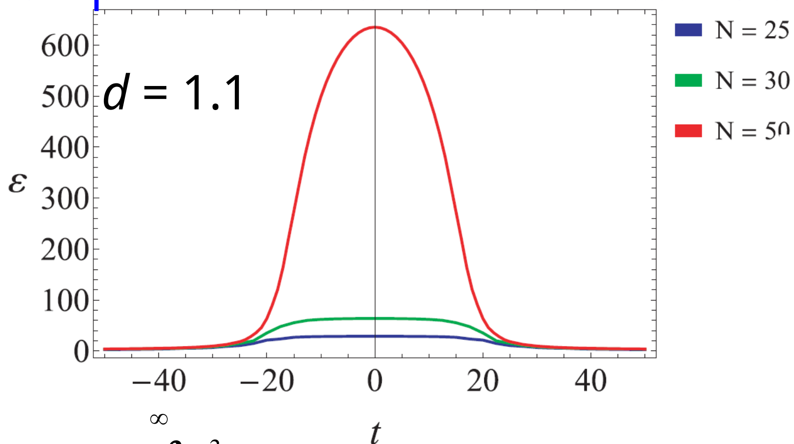
$$\varepsilon = \int_{-\infty}^{\infty} u^3\,dx \Big/ \int_{-\infty}^{\infty} (u_x)^2\,dx$$

For asymptotically large times $t \to \infty$ when solitons are sparse, the integrals in the relation between I_n and μ_n can be calculated exactly:

$$I_n = 24^n \frac{[(n-1)!]^2}{(2n-1)!} N\langle k^{2n-1}\rangle \qquad \mu_n(\infty) = 2^{3n-1}\frac{[(n-1)!]^2}{(2n-1)!} N\langle k^{2n-1}\rangle$$

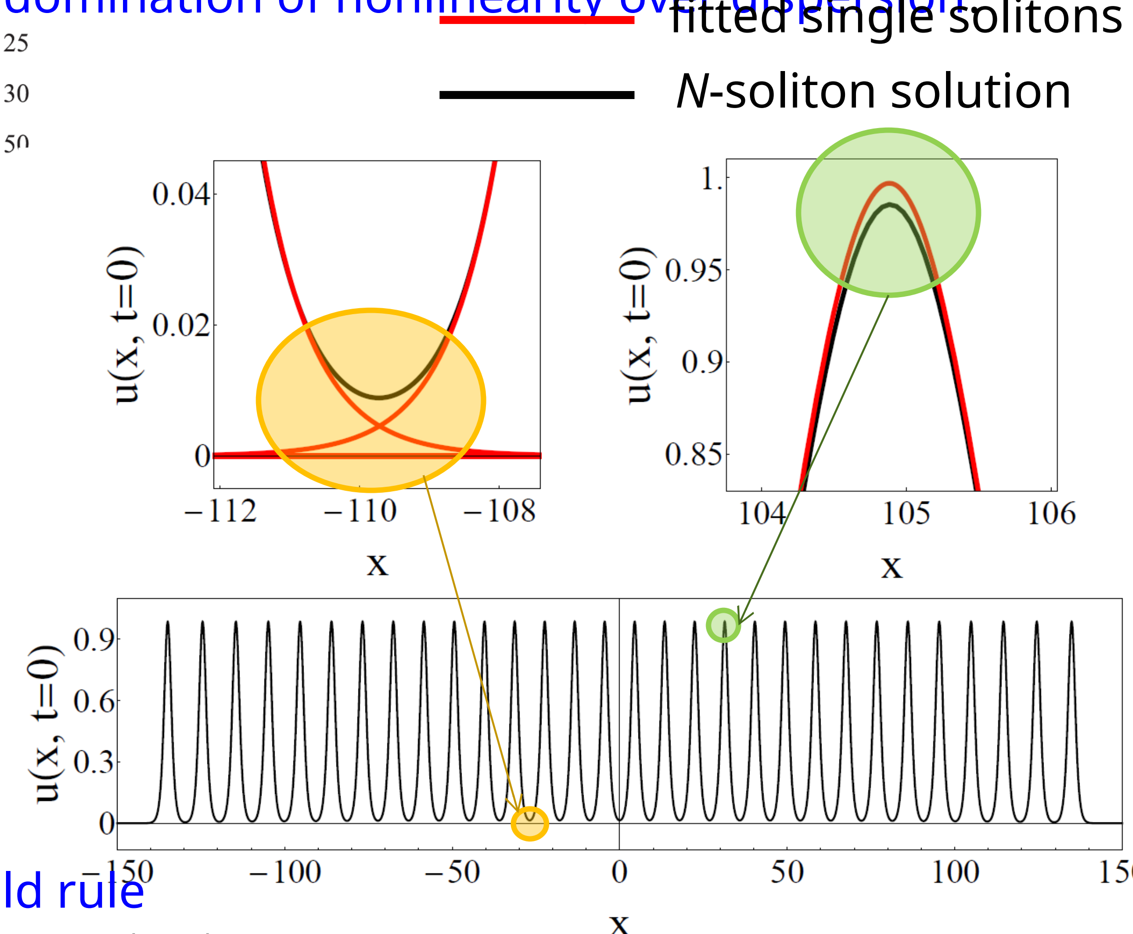
Reduction of statistical moments

It is found that the similarity parameter grows greatly when *d* is close to 1 and the number of solitons *N* increases: a collision of many solitons with slowly decaying amplitudes leads to the effective domination of nonlinearity over dispersion.



$$\varepsilon = \frac{\int_{-\infty}^{\infty} u^3\,dx}{\int_{-\infty}^{\infty} (u_x)^2\,dx} \propto AL^2$$

$$Ur^2 = \left(\int_{-\infty}^{\infty} \sqrt{u}\,dx \right)^2 \propto AL^2$$



d = 1.001

According to the Bohr-Sommerfeld rule *Ur* ~*N*, it should be approximately constant in the course of evolution. In our case the quantity for any number of isolated solitons is [illegible](*t*[illegible]) [illegible] 2: average or spectral similarity parameter.

Reduction of statistical moments

It is found that the similarity parameter grows greatly when *d* is close to 1 and the number of solitons *N* increases: a collision of many solitons with slowly decaying amplitudes leads to the effective domination of nonlinearity over dispersion.

Then the ratio of the statistical moments near the focal point $t \to 0$ and at large times reads

$$\frac{\mu_n(0)}{\mu_n(\infty)} = \frac{n I_n}{6^n \mu_n(\infty)}\left(1+O\left(\varepsilon^{-1}\right)\right) = \frac{2n}{2^n}\left(1+O\left(\varepsilon^{-1}\right)\right) \approx \frac{2n}{2^n}, \quad n = 1,2,\ldots$$

This estimation is valid when ε >> 1 for any number *n*.

When *n* =1 or *n* = 2 the equality is identical due to the proportionality between the moments and the conserved quantities: $\mu_1 \sim I_1$, $\mu_2 \sim I_2$:

$$\frac{\mu_1(0)}{\mu_1(\infty)} = 1 \qquad \frac{\mu_2(0)}{\mu_2(\infty)} = 1$$

The estimation is very accurate even for large orders of statistical moments. All the moments decrease when solitons interact. The decrease is greater for larger orders *n*.

The minimum relative values of the nth statistical moments $M_n(0)/M_n(\infty)$. The numerical solution corresponds to the case $d = 1.1$ and $N = 50$.

n	Analytical estimate	Numerical solution	Difference
3	3/4	0.7506	0.08%
4	1/2	0.5011	0.22%
5	5/16	0.3138	0.42%
6	3/16	0.1889	0.72%
7	7/64	0.1106	1.1%

Broader generality of the result

The modified KdV equation possesses an infinite set of conserved quantities too:

$$\frac{\partial w}{\partial t} + 6w^2 \frac{\partial w}{\partial x} + \frac{\partial^3 w}{\partial x^3} = 0 \qquad \frac{d}{dt} I_n = 0, \quad I_n = \int_{-\infty}^{\infty} T_n(w)\,dx, \quad n = 1,2,\ldots$$

$$I_1 = \int_{-\infty}^{\infty} w\,dx \qquad I_2 = \int_{-\infty}^{\infty} w^2\,dx \qquad I_3 = \int_{-\infty}^{\infty} \left(w^4 - (w_x)^2\right)dx \qquad I_4 = \ldots$$

Following the same way, the statistical moments of even orders can be related to the conserved integrals:

$$I_m^{(mKdV)} \propto \mu_{2(m-1)}^{(mKdV)} \left(1 + O(\delta^{-2})\right), \quad m = 2,3,\ldots$$

Here $\mu_n^{(mKdV)} = \int_{-\infty}^{\infty} w^n\,dx$

and δ^2 is the new similarity parameter $\delta^2 = \int_{-\infty}^{\infty} w^4\,dx \Big/ \int_{-\infty}^{\infty} (w_x)^2\,dx$

This consideration yields exactly the same result as before:

$$\frac{\mu_n^{(mKdV)}(0)}{\mu_n^{(mKdV)}(\infty)} = \frac{2n}{2^n}\left(1 + O(\delta^{-2})\right) \approx \frac{2n}{2^n}, \quad n = 1,2,\ldots$$

which turns out to be correct to odd orders of the moments too (checked numerically).

Broader generality of the result

The complex KdV equation for $q(x,t) \in C$

$$\frac{\partial q}{\partial t} + 6q\frac{\partial q}{\partial x} + \frac{\partial^3 q}{\partial x^3} = 0$$

is related to the mKdV equation through the complex Miura transformation

$$q(x,t) = w^2 - i\frac{\partial w}{\partial x}$$

The balance of terms in the Miura transformation is controlled by the same similarity parameter as in the mKdV framework. For [illegible] >> 1 the real term dominates over the imaginary one, and then

$$\int_{-\infty}^{\infty} q^n(x,0)\,dx \approx \int_{-\infty}^{\infty} w^{2n}(x,0)\,dx$$

At asymptotically large times solitons of the mKdV equation are mapped to the solitary solutions of the complex KdV equation which has exactly the same functional form as the solutions of the classic KdV except an imaginary phase shift.

These reasoning helps to obtain the relation for moments of the solution of the complex KdV equation:

$$\frac{\mu_n^{(cKdV)}(0)}{\mu_n^{(cKdV)}(\infty)} \approx 2^{n-1}\frac{\mu_{2n}^{(mKdV)}(0)}{\mu_{2n}^{(mKdV)}(\infty)} \approx \frac{2n}{2^n}, \quad n = 1,2,\ldots$$

Broader generality of the result

- For all members of the hierarchy of integrable KdV equations

$$\begin{cases} -\psi_{xx} + u\psi = \lambda\psi \\ \psi_t = \hat{A}\psi \end{cases}$$

the N-soliton solutions may be constructed using the Darboux transform [Matveev & Salle, 1991]. The difference will be in the time dependence of solitons only (i.e., other expressions for the soliton velocity as a function of the spectral parameter $V_j(k_j)$), but this difference vanishes at the focusing moment $t = 0$ and has no effect on the statistical moments when the solitons separate at asymptotically large times.

Consequently, the moments $\mu_n(0)$ and $\mu_n(\Box)$ will have exactly the same values as before and the estimations for the statistical moment drops remain.

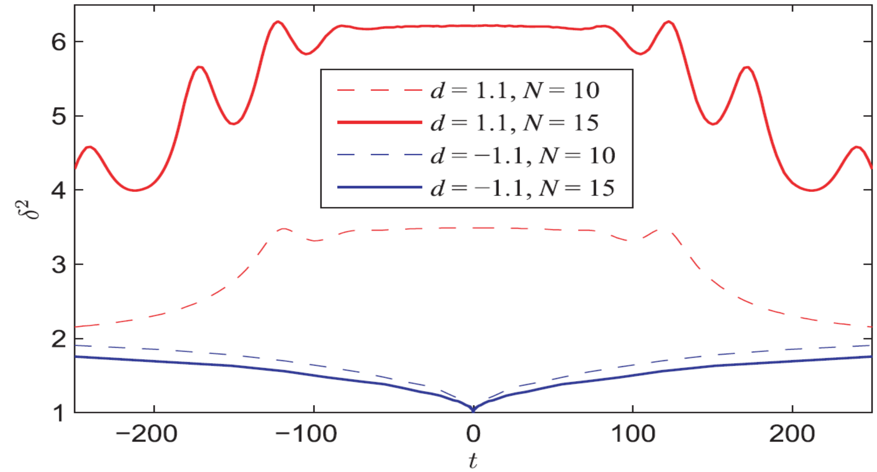
- The generalization to planar soliton solutions of integrable extensions in higher dimensions is clear too.

- Some different distributions of soliton amplitudes will obvious result in qualitatively the same behavior.

- It is quite natural to expect qualitatively similar results in non-integrable

Trivial or surprising results?

When dealing with solitons of alternating signs (consider the mKdV framework), the similarity parameter in the course of interaction decreases from $\delta^2 = 2$ (characterizes single soliton solutions) down to $\delta^2 = 1$. Therefore at the moment of generation of extremely large waves the similarity parameter is neither large nor small.



A great number of interacting solitons of the same sign produce a strongly nonlinear state in terms of the similarity parameter δ^2 (zero-dispersion limit in some sense), but do not lead to the generation of high waves. The quantity of soliton density is limited from above in this case.

At the same time, collisions of solitons of different signs may cause extremely high wave amplitudes but are characterized by a relatively small ratio of nonlinearity vs dispersion.

<u>The work is published:</u>

T.V. Tarasova, A.V. Slunyaev, Properties of synchronous collisions of solitons in the Korteweg – de Vries equation. *Communications in Nonlinear Science and Numerical Simulation* (In Press, 2022).
doi: 10.1016/j.cnsns.2022.107048

A.V. Slunyaev, T.V. Tarasova, Statistical properties of extreme soliton collisions. *Chaos* 32,