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ON A DISTRIBUTED NONLINEAR DYNAMICAL SYSTEM IN THE SPACE OF DOUBLE-SIDED SEQUENCES

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Space $\mathbb{L}$ of the complex-valued double-sided sequences

Let $u = (\ldots, u_{-n}, \ldots, u_{-1}, u_0, u_1, \ldots, u_n, \ldots)$ be the complex-valued double-sided sequence. We shall say that u is an element of the linear space of complex-valued double-sided sequences $\mathbb{L}$ if for all $z \in K$, where $K = \{z \in \mathbb{C} \mid r <| z |< R\}$ is some open ring, series $\sum_{n=-\infty}^{n=+\infty} u_n z^n$ is the Laurent series for some analytical on K function $U(z)$. Analytical on the ring K function $U(z) = \sum_{n=-\infty}^{+\infty} u_n z^n$ is called generating function for vector $u \in \mathbb{L}$. If $u \in \mathbb{L}$ and $v \in \mathbb{L}$ then:

$$U(z)\, V(z) = \sum_{n=-\infty}^{+\infty} (u \star v)_n\, z^n\,,$$

where $u \star v$ denotes double-sided sequence with components:

$$(u \star v)_n \;=\; \sum_{k=-\infty}^{+\infty} u_k\, v_{n-k}\,, \qquad n \in \mathbb{Z}\,.$$

It means that one can define finite product of two vectors from $\mathbb{L}$ without usage of generating functions.

Genesis of space $\mathbb{L}$:

$$w_t + w\, w_x = w_{xx}\,, \qquad w(x + 2\,\pi, 0) = w(x, 0)\,.$$

$$w(x, t) = \sum_{n=-\infty}^{+\infty} w_n(t)\, \exp(-i\, n\, x)\,.$$

$$\dot{w}_n = -n^2\, w_n + i \sum_{k=-\infty}^{+\infty} k\, w_k\, w_{n-k}\,.$$

Elimination of nonhomogeneity on k.

$$\dot{u} = u - u \star u\,.$$

$u(t)$ is the curve in space $\mathbb{L}$.

How to construct a surface in space $\mathbb{L}$?

Let us consider the following nonlinear countable-dimensional system of integro-differential equations:

$$\frac{\partial u_n(x,t)}{\partial t} + \sum_{k=-\infty}^{+\infty} \int_{-\infty}^{+\infty} u_k(x-\xi,t)\, u_{n-k}(\xi,t)\, d\xi = 0\,, \qquad (1)$$

where $\{u_n(x,t)\}_{n=-\infty}^{n=+\infty}$ is denumerable set of unknown functions. System (1) ought to be provided by the next denumerable set of initial conditions:

$$u_n(x,0) = u_n^0(x)\,, \qquad x \in \mathbb{R}\,. \qquad (2)$$

One can rewrite system (1) as dynamical system in $\mathbb{L}$:

$$\frac{\partial u(x,t)}{\partial t} = -\int_{-\infty}^{+\infty} u(x-\xi,t) \star u(\xi,t)\, d\xi\,, \qquad u(x,t) \in \mathbb{L}\,. \quad (3)$$

Theorem 1.

General representation of exact solution of the Cauchy problem (1)-(2) is equal to:

$$u_n(x,t) = \oint_{C_\rho} \int_{-\infty}^{+\infty} \frac{\tilde{U}^0(z;k)}{1 + t\,\tilde{U}^0(z;k)} \, \frac{\exp(i\,k\,x)}{z^{n+1}} \, \frac{dk}{2\,\pi} \, \frac{dz}{2\,\pi\,i} \qquad (4)$$

where

$$\tilde{U}^0(z;k) = \int_{-\infty}^{+\infty} U^0(z;x) \exp(-i\,k\,x) dx \qquad (5)$$

is the Fourier transform from the generating function of its initial condition:

$$U^0(z;x) = \sum_{n=-\infty}^{+\infty} u_n^0(x)\, z^n\,, \qquad (6)$$

integration along the circle $C_\rho = \{z \in \mathrm{C} \mid \mid z \mid = \rho\}$ being counter clockwise.

Sketch of the proof:

$$U(z;x,t) = \sum_{n=-\infty}^{+\infty} u_n(x,t)\, z^n\,, \qquad U(z;x,0) = U^0(z;x)\,.$$

$$\frac{\partial U(z;x,t)}{\partial t} + \int_{-\infty}^{+\infty} U(z;x-\xi,t)\, U(z;\xi,t) d\xi = 0\,.$$

$$\tilde{U}(z;k,t) = \int_{-\infty}^{+\infty} U(z;x,t) \exp(-i\,k\,x) dx\,.$$

$$\frac{\partial \tilde{U}(z;k,t)}{\partial t} + \tilde{U}^2(z;k,t) = 0\,, \qquad \tilde{U}(z;k,0) = \tilde{U}^0(z;k)\,.$$

$$\tilde{U}(z;k,t) = \frac{\tilde{U}^0(z;k)}{1 + t\,\tilde{U}^0(z;k)}\,.$$

$$U(z;x,t) = \int_{-\infty}^{+\infty} \frac{\tilde{U}^0(z;k)}{1 + t\,\tilde{U}^0(z;k)}\, \exp(i\,k\,x)\, \frac{dk}{2\,\pi}\,.$$

Exact solution with oscillatory behavior

Let us consider the following vector $\{u_n^0(x)\}_{n=-\infty}^{n=+\infty}$ of initial conditions:

$$u_n^0(x) = (-1)^n \frac{A_0}{4\,a_0} \left[J_{n+1}\left(\frac{|x|}{a_0}\right) - J_{n-1}\left(\frac{|x|}{a_0}\right)\right], \quad n \in \mathbb{N}\,,$$

$$u_0^0(x) = \frac{A_0}{2\,a_0}\, J_1\left(\frac{|x|}{a_0}\right), \quad u_{-n}^0(x) = (-1)^n u_n^0(x), \tag{7}$$

where A_0, $a_0 > 0$ and $J_n(\zeta)$ are Bessel functions of the first kind:

$$\exp\left[\frac{\zeta}{2}\left(z - \frac{1}{z}\right)\right] = \sum_{n=-\infty}^{+\infty} J_n(\zeta)\, z^n\,.$$

Initial conditions with oscillatory behavior:

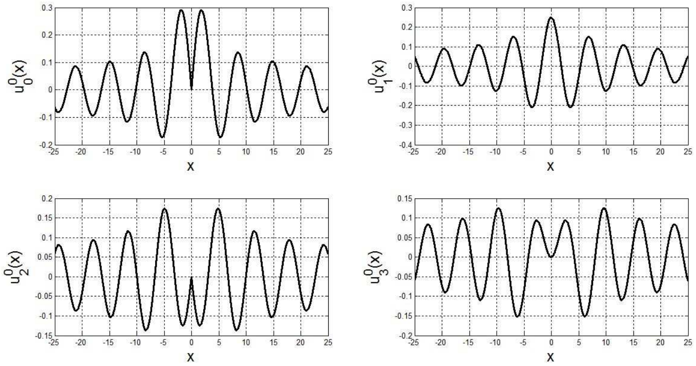

Рис.: Graphs of the first functions $u_n^0(x)$ under $A_0 = 1$ and $a_0 = 1$

Theorem 2.

Exact solution of the Cauchy problem (1)-(2) with initial conditions (7) is equal to ($n \in \mathbb{N}$):

$$u_n(x,t) = \frac{(-1)^n A_0}{4 a_0 \sqrt{1 + A_0 t}} \left[J_{n+1}\left(\frac{\sqrt{1 + A_0 t}|x|}{a_0}\right) - J_{n-1}\left(\frac{\sqrt{1 + A_0 t}|x|}{a_0}\right)\right]$$

$$u_0(x,t) = \frac{A_0}{2\, a_0 \sqrt{1 + A_0 t}}\, J_1\left(\frac{\sqrt{1 + A_0 t}\,|x|}{a_0}\right),$$

$$u_{-n}(x,t) = (-1)^n u_n(x,t)\,. \qquad (8)$$

Components of exact solution with oscillatory behavior:

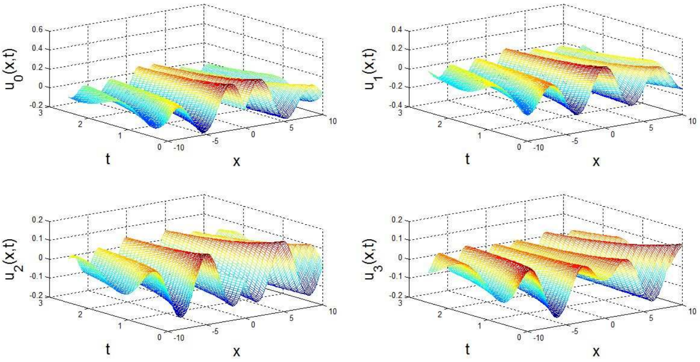

Рис.: Spatiotemporal evolution of the first functions $u_n(x,t)$ under $A_0 = 1$ and $a_0 = 1$

Sketch of the proof:

$$U^0(z;x) = \frac{A_0}{4\,a_0}\left(z - \frac{1}{z}\right)\exp\left[-\frac{|x|}{2\,a_0}\left(z - \frac{1}{z}\right)\right].$$

$$\tilde{U}^0(z;k) = A_0\left[1 + k^2\left(\frac{2\,a_0\,z}{z^2-1}\right)^2\right]^{-1}.$$

$$\tilde{U}(z;k,t) = A_0\left[1 + A_0\,t + k^2\left(\frac{2\,a_0\,z}{z^2-1}\right)^2\right]^{-1}.$$

$$U(z;x,t) = \frac{A_0}{4\,a_0\sqrt{1+A_0\,t}}\left(z - \frac{1}{z}\right)\exp\left[-\frac{\sqrt{1+A_0\,t}\,|x|}{2\,a_0}\left(z - \frac{1}{z}\right)\right].$$

Exact solution with monotone behavior

Let us consider the following vector $\{u_n^0(x)\}_{n=-\infty}^{n=+\infty}$ of initial conditions:

$$u_n^0(x) = (-1)^{n+1}\frac{A_0}{4\,a_0}\left[I_{n+1}\left(\frac{|x|}{a_0}\right) + I_{n-1}\left(\frac{|x|}{a_0}\right)\right], \quad n \in \mathbb{N},$$

$$u_0^0(x) = -\frac{A_0}{2\,a_0}\,I_1\left(\frac{|x|}{a_0}\right), \quad u_{-n}^0(x) = u_n^0(x)\,, \tag{9}$$

where A_0, $a_0 > 0$ and $I_n(\zeta)$ are modified Bessel functions:

$$\exp\left[\frac{\zeta}{2}\left(z + \frac{1}{z}\right)\right] = \sum_{n=-\infty}^{+\infty} I_n(\zeta)\,z^n\,.$$

Initial conditions with monotone behavior:

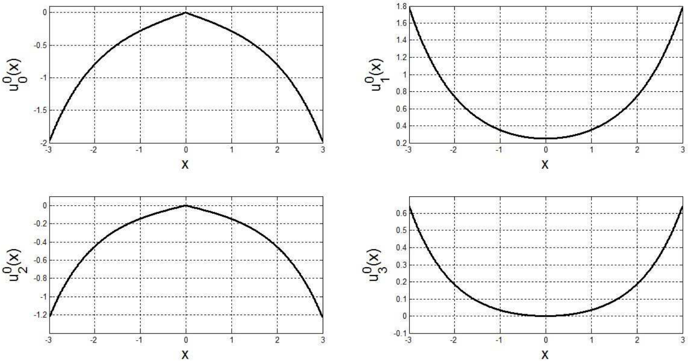

Рис.: Graphs of the first functions $u_n^0(x)$ under $A_0 = 1$ and $a_0 = 1$

Theorem 3.

Exact solution of the Cauchy problem (1)-(2) with initial conditions (9) is equal to ($n \in \mathbb{N}$):

$$u_n(x,t) = \frac{(-1)^{n+1} A_0}{4 a_0 \sqrt{1 + A_0\, t}} \left[I_{n+1}\left(\frac{\sqrt{1 + A_0\, t}|x|}{a_0}\right) + I_{n-1}\left(\frac{\sqrt{1 + A_0\, t}|x|}{a_0}\right)\right]$$

$$u_0(x,t) = -\frac{A_0}{2\, a_0 \sqrt{1 + A_0\, t}}\, I_1\left(\frac{\sqrt{1 + A_0\, t}\,|x|}{a_0}\right),$$

$$u_{-n}(x,t) = u_n(x,t). \qquad (10)$$

Components of exact solution with monotone behavior:

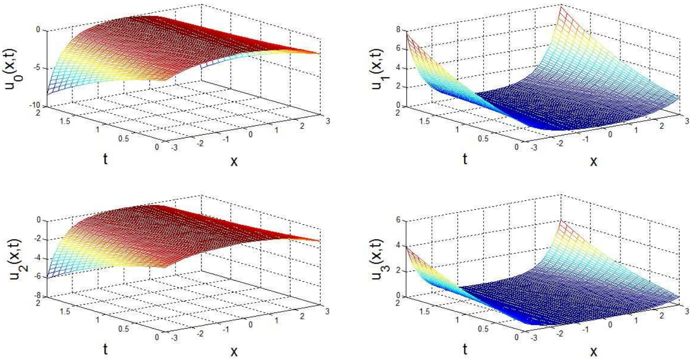

Рис.: Spatiotemporal evolution of the first functions $u_n(x,t)$ under $A_0 = 1$ and $a_0 = 1$

Concluding remarks

If generating function $U(z;x,t)$ represents exact solution of the Cauchy problem (1)-(2) with initial condition representing by generating function $U^0(z;x)$ then for any $p = 2,3,4,\ldots$ generating function $U(z^p;x,t)$ represents exact solution of the Cauchy problem (1)-(2) with initial condition representing by generating function $U^0(z^p;x)$. In other words the Laurent expansion for transformed generating function $U(z^p;x,t)$ gives one exact solution $\{\hat{u}_m(x,t)\}_{m=-\infty}^{m=+\infty}$ of the Cauchy problem (1)-(2) too as follows:

$$\hat{u}_{np}(x,t) = u_n(x,t)\,, \qquad n \in \mathbb{Z}\,, \tag{11}$$

where $u_n(x,t)$ are functions (8) or (10), and place between components of double-sided vector $\{\hat{u}_m(x,t)\}_{m=-\infty}^{m=+\infty}$ with numbers $n\,p$ and $n\,p+p$ are filled by zeros. The initial condition $\{\hat{u}^0_m(x)\}_{m=-\infty}^{m=+\infty}$ in this case has the same structure as the formulas (11).

At last let us consider the following nonlinear integro-differential equation:

$$\frac{\partial u(x,y,t)}{\partial t} + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u(x-\xi, y-\eta, t)\, u(\xi,\eta,t)\, d\xi\, d\eta = 0\,,$$

provided by initial condition:

$$u(x,y,0) = u^0(x,y)\,, \qquad (x,y) \in \mathbb{R}^2.$$

It is easy to see that the Cauchy problem (1)-(2) arises from this Cauchy problem .

THANK YOU FOR YOUR ATTENTION!