Inverse cascade spectrum of surface gravity waves in the presence of condensate: analytical explanation.

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Zakharov-Kolmogorov solutions (3D fluid, 2D surface, gravity waves, deep water)

Direct cascade of energy from large scale to small scales (Zakharov and Filonenko, 1967)

$$n_k^{(1)} = C_1 P^{1/3} k^{-\frac{2\beta}{3}-d} = C_1 P^{1/3} k^{-4}. \quad (1)$$

(Proven in previous works).
Inverse cascade of wave action or number of waves (Zakharov and Zaslavsky, 1988)

$$n_k^{(2)} = C_2 Q^{1/3} k^{-\frac{2\beta-\alpha}{3}-d} = C_2 Q^{1/3} k^{-23/6} \approx const \;\; k^{-3.83}. \quad (2)$$

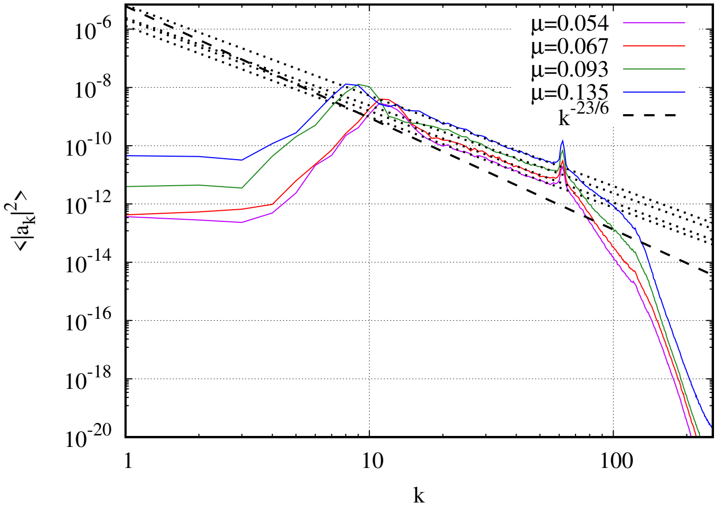
Numerical scheme parameters

Let us add damping and pumping in dynamical equations

$$\begin{aligned}
\dot\eta &= \hat k\psi - (\nabla(\eta\nabla\psi)) - \hat k[\eta\hat k\psi] + \\
&\quad + \hat k(\eta\hat k[\eta\hat k\psi]) + \frac{1}{2}\Delta[\eta^2\hat k\psi] + \frac{1}{2}\hat k[\eta^2\Delta\psi] - F^{-1}[\gamma_k\eta_{\vec k}], \\
\dot\psi &= -g\eta - \frac{1}{2}\left[(\nabla\psi)^2 - (\hat k\psi)^2\right] - \\
&\quad - [\hat k\psi]\hat k[\eta\hat k\psi] - [\eta\hat k\psi]\Delta\psi - F^{-1}[\gamma_k\psi_{\vec k}] + F^{-1}[f_k \mathrm{e}^{\mathrm{i}R_{\vec k}(t)}], \\
f_k &= 4F_0\frac{(k-k_{p1})(k_{p2}-k)}{(k_{p2}-k_{p1})^2}; \\
D_{\vec k} &= \gamma_k\psi_{\vec k}, \quad \gamma_k = \gamma_0(k-k_d)^2, k > k_d.
\end{aligned}$$

Here $R_{\vec k}(t)$ — uniformly distributed random number in interval $(0, 2\pi]$. Simulation region $L_x = L_y = 2\pi$ with double periodic boundary conditions. Grid resolution $N_x = N_y = 512$. Pumping parameters: $F_0 = 5\times 10^{-9}(\times 2, \times 4, \times 8), k_{p1} = 60, k_{p2} = 64$. Damping starts at $k_d = 128$.

Spectra. Angle averaged. $t \simeq 10^6 T_p$.

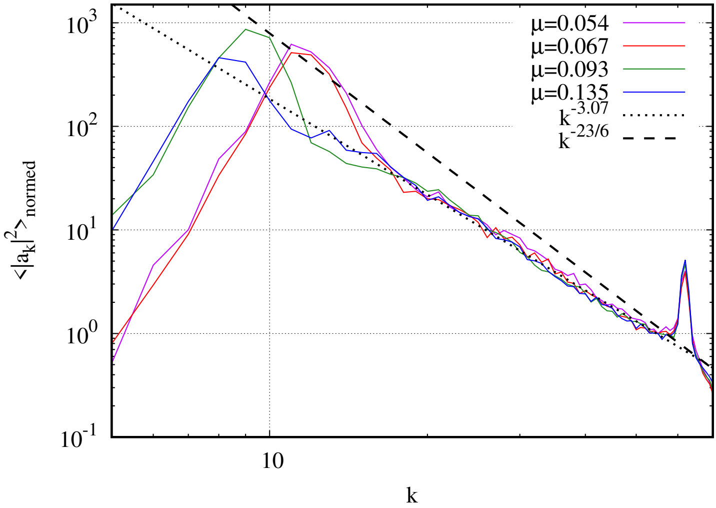


Least squares fit: slopes for angle averaged spectra.

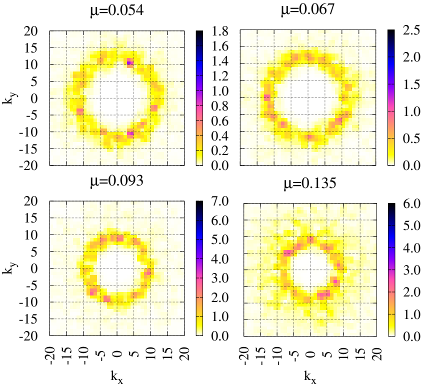
μ	$k \in$	Average slope	Slope error
0.054	$[17; 55]$	-3.12	± 0.04
0.067	$[16; 55]$	-3.14	± 0.05
0.093	$[12; 56]$	-3.01	± 0.05
0.135	$[11; 56]$	-3.11	± 0.04
All	170 points	-3.07	± 0.02

Table: Least squares fits for different simulation spectra. The second column shows the range of k between the condensate and pumping influence regions; the third column gives average slope α for $\langle |a_k|^2 \rangle \sim k^{\alpha}$; the last column give an estimated error of the fit.

Spectra. Angle averaged and normed. $t \simeq 10^6 T_p$.



How condensate looks like: random, not coherent.



Kinetic equation with condensate.

Looking for constant flux solution for wave action N_k.

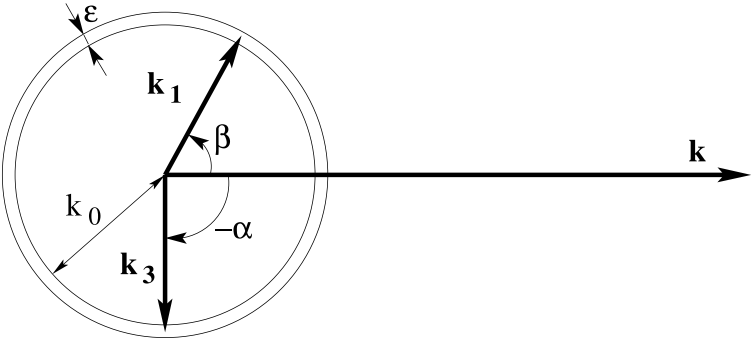
$$\frac{\partial N_{\vec{k}}}{\partial t} = \int \left| T_{\vec{k},\vec{k}_1,\vec{k}_2,\vec{k}_3} \right|^2 N_{\vec{k}} N_{\vec{k}_1} N_{\vec{k}_2} N_{\vec{k}_3} \left(\frac{1}{N_{\vec{k}}} + \frac{1}{N_{\vec{k}_1}} - \frac{1}{N_{\vec{k}_2}} - \frac{1}{N_{\vec{k}_3}} \right) \times$$
$$\times \delta(\vec{k} + \vec{k}_1 - \vec{k}_2 - \vec{k}_3) \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \mathrm{d}\vec{k}_1 \mathrm{d}\vec{k}_2 \mathrm{d}\vec{k}_3.$$

Because condensate is bigger than anything, two waves $\vec{k}_1$ and $\vec{k}_3$ are in condensate:

$$\vec{k} + \vec{k}_1 = \vec{k}_2 + \vec{k}_3, \;\; \omega_k + \omega_{k_1} = \omega_{k_2} + \omega_{k_3}.$$

Vectors. Schematics.

Everything is isotropic wrt polar angles:



$\Delta\vec{k} = \vec{k}_1 - \vec{k}_3$, $\vec{k}_2 = \vec{k} + \Delta\vec{k}$, $\omega_k + \omega_{k_1} = \omega_{k_2} + \omega_{k_3}$.
Let us consider $k \gg k_0$, so k_0/k is a small parameter.

$N_{\vec{k}_1}, N_{\vec{k}_3} \gg N_{\vec{k}}, N_{\vec{k}+\Delta\vec{k}}.$

$$\frac{\partial N_{\vec{k}}}{\partial t} \approx \int \left| T_{\vec{k},\vec{k}_1,\vec{k}+\Delta\vec{k},\vec{k}_3} \right|^2 N_{\vec{k}} N_{\vec{k}_1} N_{\vec{k}+\Delta\vec{k}} N_{\vec{k}_3} \left(\frac{1}{N_{\vec{k}}} - \frac{1}{N_{\vec{k}+\Delta\vec{k}}} \right) \times$$
$$\times \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \mathrm{d}\vec{k}_1 \mathrm{d}\vec{k}_3 =$$
$$= \int \left| T_{\vec{k},\vec{k}_1,\vec{k}+\Delta\vec{k},\vec{k}_3} \right|^2 N_{\vec{k}_1} N_{\vec{k}_3} (N_{\vec{k}+\Delta\vec{k}} - N_{\vec{k}}) \times$$
$$\times \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \mathrm{d}\vec{k}_1 \mathrm{d}\vec{k}_3.$$

Using small parameter k_0/k and choosing ε in the right way:

$$\delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \approx \delta(\omega_{k_1} - \omega_{k_3}).$$

$$N_{\vec{k}+\Delta\vec{k}} - N_{\vec{k}} \approx \Delta\vec{k}\cdot\vec{\nabla}_{\vec{k}} N_{\vec{k}},$$

Exchange dummy variables $\vec{k}_1$ and $\vec{k}_3$ (requires $\Delta k \to -\Delta k$):

$$\frac{\partial N_{\vec{k}}}{\partial t} \approx \frac{1}{2}\int \left| T_{\vec{k},\vec{k}_1,\vec{k}+\Delta\vec{k},\vec{k}_3}\right|^2 N_{\vec{k}_1} N_{\vec{k}_3} \Delta\vec{k}\cdot\vec{\nabla}_{\vec{k}} N_{\vec{k}} \delta(\omega_{k_1} - \omega_{k_3}) \mathrm{d}\vec{k}_1 \mathrm{d}\vec{k}_3 + \\ + \frac{1}{2}\int \left| T_{\vec{k},\vec{k}_3,\vec{k}-\Delta\vec{k},\vec{k}_1}\right|^2 N_{\vec{k}_1} N_{\vec{k}_3} (-\Delta\vec{k})\cdot\vec{\nabla}_{\vec{k}} N_{\vec{k}} (\omega_{k_1} - \omega_{k_3}) \mathrm{d}\vec{k}_1 \mathrm{d}\vec{k}_3.$$

Let us introduce function:

$$G(\vec{k},\vec{k}_1,\vec{k}_3) = \left| T_{\vec{k},\vec{k}_1,\vec{k}+\Delta\vec{k},\vec{k}_3}\right|^2 \Delta\vec{k}\cdot\vec{\nabla}_{\vec{k}} N_{\vec{k}}.$$

Using symmetry $T_{\vec{k},\vec{k}_1,\vec{k}_2,\vec{k}_3} = T_{\vec{k}_2,\vec{k}_3,\vec{k},\vec{k}_1}$

$$\frac{\partial N_{\vec{k}}}{\partial t} \approx \frac{1}{2}\int N_{\vec{k}_1}N_{\vec{k}_3}(G(\vec{k},\vec{k}_1,\vec{k}_3) - G(\vec{k}-\Delta\vec{k},\vec{k}_1,\vec{k}_3))\delta(\omega_{k_1}-\omega_{k_3})\mathrm{d}\vec{k}_1\mathrm{d}\vec{k}_3 \approx$$
$$\approx \frac{1}{2}\int N_{\vec{k}_1}N_{\vec{k}_3}(\Delta\vec{k}\cdot\vec{\nabla}_{\vec{k}})G(\vec{k},\vec{k}_1,\vec{k}_3)\delta(\omega_{k_1}-\omega_{k_3})\mathrm{d}\vec{k}_1\mathrm{d}\vec{k}_3 =$$
$$= \vec{\nabla}_{\vec{k}}\cdot\frac{1}{2}\int N_{\vec{k}_1}N_{\vec{k}_3}\Delta\vec{k}\left(\left|T_{\vec{k},\vec{k}_1,\vec{k},\vec{k}_3}\right|^2\frac{\Delta\vec{k}\cdot\vec{k}}{k}N'_{\vec{k}}\right)\delta(\omega_{k_1}-\omega_{k_3})\mathrm{d}\vec{k}_1\mathrm{d}\vec{k}_3$$

This is continuity equation for wave action:

$$\frac{\partial N_{\vec{k}}}{\partial t} = -\vec{\nabla}_{\vec{k}}\cdot\vec{Q}_{\vec{k}}.$$

If everything is isotropic:

$$N = \int n_{\vec{k}}\mathrm{d}\vec{k} = \int n_k 2\pi k\mathrm{d}k,\quad \vec{Q}_{\vec{k}} = \frac{\vec{k}}{k}Q_k,\quad \frac{\partial 2\pi k N_k}{\partial t} = -\frac{\partial 2\pi\vec{k}\cdot\vec{Q}_{\vec{k}}}{\partial k}.$$

$$Q = 2\pi \vec{k} \cdot \vec{Q}_k \quad = \left[\pi \int N_{\vec{k}_1} N_{\vec{k}_3} \delta(\omega_{k_1} - \omega_{k_3}) \left(\left| T_{\vec{k},\vec{k}_1,\vec{k},\vec{k}_3} \right|^2 (\Delta \vec{k} \cdot \vec{k})^2 \right) \times \right.$$
$$\left. \times \delta(\omega_{k_1} - \omega_{k_3}) \mathrm{d}\vec{k}_1 \mathrm{d}\vec{k}_3 \right] N'_k.$$

Thus, we have diffusion equation with inhomogeneous diffusion coefficient D_k:

$$2\pi k \frac{\partial N_{\vec{k}}}{\partial t} = \frac{\partial}{\partial k} D_k \frac{\partial N_k}{\partial k}.$$

$$D_k = \pi k \int\limits_{k_1 \ll k} N_{k_1}^2 \frac{k_1^4}{\left| \frac{\partial \omega_{k_1}}{\partial k_1} \right|} \left[\int\limits_0^{2\pi} \int\limits_0^{2\pi} \left| T_{\vec{k},\vec{k}_1,\vec{k},\vec{k}_3} \right|^2 (\cos\beta - \cos\alpha)^2 \mathrm{d}\alpha \mathrm{d}\beta \right] \mathrm{d}k_1.$$

$$
\begin{aligned}
T_{1234} &= \frac{1}{2}(\tilde{T}_{1234} + \tilde{T}_{2134}),\\
\tilde{T}_{1234} &= -\frac{1}{16\pi^2}\frac{1}{(k_1 k_2 k_3 k_4)^{1/4}}\\
&\times \Big\{ -12 k_1 k_2 k_3 k_4 - 2(\omega_1+\omega_2)^2 [\omega_3\omega_4((\vec{k}_1\cdot\vec{k}_2) - k_1 k_2)\\
&+ \omega_1\omega_2((\vec{k}_3\cdot\vec{k}_4) - k_3 k_4)]\frac{1}{g^2}\\
&- 2(\omega_1-\omega_3)^2[\omega_2\omega_4((\vec{k}_1\cdot\vec{k}_3) + k_1 k_3) + \omega_1\omega_3((\vec{k}_2\cdot\vec{k}_4) + k_2 k_4)]\frac{1}{g^2}\\
&- 2(\omega_1-\omega_4)^2[\omega_2\omega_3((\vec{k}_1\cdot\vec{k}_4) + k_1 k_4) + \omega_1\omega_4((\vec{k}_2\cdot\vec{k}_3) + k_2 k_3)]\frac{1}{g^2}\\
&+ [(\vec{k}_1\cdot\vec{k}_2) + k_1 k_2][(\vec{k}_3\cdot\vec{k}_4) + k_3 k_4]\\
&+ [-(\vec{k}_1\cdot\vec{k}_3) + k_1 k_3][-(\vec{k}_2\cdot\vec{k}_4) + k_2 k_4]\\
&+ [-(\vec{k}_1\cdot\vec{k}_4) + k_1 k_4][-(\vec{k}_2\cdot\vec{k}_3) + k_2 k_3] +
\end{aligned} \tag{3}
$$

$$+4(\omega_1+\omega_2)^2 \frac{[(\vec{k}_1\cdot\vec{k}_2)-k_1k_2][-(\vec{k}_3\cdot\vec{k}_4)-k_3k_4]}{\omega_{1+2}-(\omega_1+\omega_2)^2}$$

$$+4(\omega_1-\omega_3)^2 \frac{[(\vec{k}_1\cdot\vec{k}_3)+k_1k_3][(\vec{k}_2\cdot\vec{k}_4)+k_2k_4]}{\omega_{1-3}-(\omega_1-\omega_3)^2}$$

$$\left. +4(\omega_1-\omega_4)^2 \frac{[(\vec{k}_1\cdot\vec{k}_4)+k_1k_4][(\vec{k}_2\cdot\vec{k}_3)+k_2k_3]}{\omega_{1-4}-(\omega_1-\omega_4)^2} \right\}.$$

(Reproduced from Pushkarev, Resio, Zakharov (2003)). Expression IS NOT symmetrized with respect to exchange of $\vec{k}_1, \vec{k}_2 \leftrightarrow \vec{k}_3, \vec{k}_4$! So symmetric matrix element is:

$$T^{sym}_{1234} = \frac{1}{2}(T_{1234} + T_{3412}).$$

The first term in small parameter k_0/k:

$$T^{sym}_{\vec{k}_1\vec{k},\vec{k}_3\vec{k}} = \frac{T(\vec{k},k_0,\alpha,\beta)+T(\vec{k},k_0,\beta,\alpha)}{2} = -\frac{(kk_0)^{3/2}}{16\pi^2}(\cos\alpha-\cos\beta)^2.$$

$$\int_0^{2\pi}\int_0^{2\pi}\left|T_{\vec{k},\vec{k}_1,\vec{k},\vec{k}_3}\right|^2(\cos\beta-\cos\alpha)^2\mathrm{d}\alpha\mathrm{d}\beta=\frac{25(kk_1)^3}{256\pi^2}.$$

Let us recall:

$$2\pi k\frac{\partial N_{\vec{k}}}{\partial t}=\frac{\partial}{\partial k}D_k\frac{\partial N_k}{\partial k}.$$

$$D_k=\pi k\int\limits_{k_1\ll k}N_{k_1}^2 k_1^{9/2}\frac{25(kk_1)^3}{256\pi^2}\mathrm{d}k_1.$$

As a result $D_k\sim k^4$, so if we look for constant flux solution:

$$\frac{\partial N_k}{\partial k}\sim\frac{const}{k^4}\Rightarrow N_k\sim k^{-3}.$$

Results.

- Performed simulation of the isotropic turbulence of gravity waves with the pumping narrow in frequency domain and significant scale available for development of inverse cascade.
- Observed formation of the inverse cascade and condensate in low frequencies.
- Currently observed slopes of the inverse cascade are close to $n_k \sim k^{-3.07}$, which differ significantly from theoretically predicted $n_k \sim k^{-23/6} \simeq k^{-3.83}$.
- Explanation through WTT theory in the presence of strong condensate is proposed: $n_k \sim k^{-3}$.